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Applications of the Complex Exponential Integral

By Murlan S. Corrington

1. Introduction. The recent publication of an extensive table of the exponential integral for complex arguments [1] makes it possible to evaluate a large number of indefinite integrals not in existing tables, and to obtain values for the sine and cosine integrals for complex arguments.

2. Definition of Exponential Integral. The definition used by the National Bureau of Standards will be used throughout,

$$E_1(z) = \int_z^\infty \frac{e^{-u}}{u}\,du = RE_1(z) + iIE_1(z) \tag{1}$$

where $z = x + iy$. R and I denote the real and imaginary components respectively. The integral converges if the upper limit is $\infty e^{i\alpha}$, and is independent of α, so long as $-\frac{1}{2}\pi \leqq \alpha \leqq \frac{1}{2}\pi$ [2]. To make $E_1(z)$ a single-valued function a branch cut is made just below the negative real axis, including the origin, such that $z = x + iy = \rho e^{i\theta}$, $-\pi < \theta \leqq \pi$. This means that when the integral is evaluated for a point on the negative real axis the contour must be indented above the pole at the origin.

The values of $E_1(z)$ are given in the tables for the region $0 < \theta \leqq \pi$; those for the region $-\pi < \theta < 0$ can be obtained from the relations

$$E_1(\bar{z}) = \overline{E_1(z)}, \qquad \bar{z} = x - iy.$$

Here, as usual, $\overline{f(z)}$ means the complex conjugate of $f(z)$.

3. Relation to Exponential Integral for Real Arguments. The earlier tables of the exponential integral [3] for real arguments use the following definitions:

$$Ei(x) = \mathcal{f}\int_{-\infty}^{x} \frac{e^u}{u}\,du = -\mathcal{f}\int_{-x}^{\infty} \frac{e^{-u}}{u}\,du \tag{2}$$

and

$$-Ei(-x) = \int_x^\infty \frac{e^{-u}}{u}\,du, \qquad x > 0 \tag{3}$$

where $\mathcal{f}$ is the Cauchy principal value. This gives the relations

$$E_1(y) = -Ei(-y), \qquad y > 0 \tag{4}$$

and

$$-E_1(-y) = Ei(y) + i\pi, \qquad y > 0 \tag{5}$$

4. Sine and Cosine Integrals for Complex Arguments. Let $u = it$, then equation (1) becomes

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$$E_1(x+iy) = \int_{y-ix}^{\infty} \frac{e^{-it}}{t}\,dt$$

$$= \int_{y-ix}^{\infty} \frac{\cos t}{t}\,dt - i\int_{y-ix}^{\infty} \frac{\sin t}{t}\,dt \tag{6}$$

$$= -Ci(y-ix) - i[\tfrac{1}{2}\pi - Si(y-ix)].$$

Interchanging letters gives the relations

$$E_1(y-ix) = -Ci(x+iy) + i[\tfrac{1}{2}\pi - Si(x+iy)] \tag{7}$$

and

$$E_1(-y+ix) = -Ci(x+iy) - i[\tfrac{1}{2}\pi - Si(x+iy)]. \tag{8}$$

Subtracting equations (7) and (8) gives

$$Si(x+iy) = \tfrac{1}{2}i\,[\overline{E_1(y+ix)} - E_1(-y+ix)] + \tfrac{1}{2}\pi$$

$$= \tfrac{1}{2}[IE_1(y+ix) + IE_1(-y+ix) + \pi] \tag{9}$$

$$+ \tfrac{1}{2}i[RE_1(y+ix) - RE_1(-y+ix)].$$

Adding equations (7) and (8) gives

$$Ci(x+iy) = -\tfrac{1}{2}[RE_1(y+ix) + RE_1(-y+ix)] \tag{10}$$

$$+ \tfrac{1}{2}i[IE_1(y+ix) - IE_1(-y+ix)].$$

In the other quadrants the following relations can be used:

$$Si(x-iy) = \overline{Si(x+iy)}, \tag{11a}$$

$$Si(-x+iy) = -\overline{Si(x+iy)}, \tag{11b}$$

$$Si(-x-iy) = Si(x+iy), \tag{11c}$$

$$Ci(x-iy) = \overline{Ci(x+iy)}, \tag{12a}$$

$$Ci(-x+iy) = \overline{Ci(x+iy)} + i\pi, \tag{12b}$$

$$Ci(-x-iy) = Ci(x+iy) - i\pi. \tag{12c}$$

When $x = 0$ in equation (6)

$$E_1(iy) = -Ci(y) - i[\tfrac{1}{2}\pi - Si(y)],$$

so

$$Si(y) = \tfrac{1}{2}\pi + IE_1(iy) \tag{13}$$

and

$$Ci(y) = -RE_1(iy). \tag{14}$$

5. Relation to Previously Published Tables. Tables of sine and cosine integrals for complex arguments in rectangular coordinates have been published by Bleick [4]. His definition of the sine integral is not the one given here, and his function is not an analytic function. Since he defined the sine integral as

$$Si_B(x+iy) = \int_{iy}^{x+iy} \frac{\sin t}{t}\,dt, \tag{15}$$

the relation to the one used here is

$$\begin{aligned} Si(x+iy) &= Si(iy) + Si_B(x+iy) \\ &= Si_B(x+iy) + \tfrac{1}{2}i[E_1(y) - E_1(-y)] + \tfrac{1}{2}\pi. \end{aligned} \tag{16}$$

His definition of the cosine integral leads to the same values as the one here.

A table of $E_1(z)$ for arguments in polar coordinates has been published by Mashiko for angles in the first quadrant [5]. Computer routines for the complex exponential integral are also available [6].

6. Table of Integrals. The following integrals, in which a and b are real, can be evaluated in terms of the real and imaginary components of the exponential integral:

$$\begin{aligned} \int_0^b \frac{\sin t}{a^2+t^2}\,dt = \frac{e^a}{2a}&[RE_1(a) - RE_1(a+ib)] \\ &- \frac{e^{-a}}{2a}[RE_1(-a) - RE_1(-a+ib)], \end{aligned} \tag{17}$$

$$\int_0^b \frac{\cos t}{a^2+t^2}\,dt = \frac{e^{-a}}{2a}[\pi + IE_1(-a+ib)] - \frac{e^a}{2a} IE_1(a+ib), \tag{18}$$

$$\int_0^b \frac{t\sin t}{a^2+t^2}\,dt = \frac{1}{2}e^{-a}[\pi + IE_1(-a+ib)] + \frac{1}{2}e^a IE_1(a+ib), \tag{19}$$

$$\begin{aligned} \int_0^b \frac{t\cos t}{a^2+t^2}\,dt = \frac{1}{2}e^a&[RE_1(a) - RE_1(a+ib)] \\ &+ \tfrac{1}{2}e^{-a}[RE_1(-a) - RE_1(-a+ib)], \end{aligned} \tag{20}$$

$$\begin{aligned} \int_0^b \frac{\sinh t}{a^2+t^2}\,dt = &-\frac{\sin a}{2a}[RE_1(b+ia) + RE_1(-b+ia) + 2Ci(a)] \\ &- \frac{\cos a}{2a}[IE_1(b+ia) + IE_1(-b+ia) - 2Si(a) + \pi], \end{aligned} \tag{21}$$

$$\begin{aligned} \int_0^b \frac{\cosh t}{a^2+t^2}\,dt = \frac{\sin a}{2a}&[RE_1(b+ia) - RE_1(-b+ia)] \\ &+ \frac{\cos a}{2a}[IE_1(b+ia) - IE_1(-b+ia)], \end{aligned} \tag{22}$$

$$\begin{aligned} \int_0^b \frac{t\sinh t}{a^2+t^2}\,dt = \frac{1}{2}\cos a&[RE_1(b+ia) - RE_1(-b+ia)] \\ &- \tfrac{1}{2}\sin a[IE_1(b+ia) - IE_1(-b+ia)], \end{aligned} \tag{23}$$

$$\begin{aligned} \int_0^b \frac{t\cosh t}{a^2+t^2}\,dt = \frac{1}{2}\sin a&[IE_1(b+ia) + IE_1(-b+ia) - 2Si(a) + \pi] \\ &- \tfrac{1}{2}\cos a[RE_1(b+ia) + RE_1(-b+ia) + 2Ci(a)], \end{aligned} \tag{24}$$

$$\begin{aligned} \int_0^b \frac{e^t}{a^2+t^2}\,dt = -\frac{\sin a}{a}&[RE_1(-b+ia) + Ci(a)] \\ &- \frac{\cos a}{a}\left[IE_1(-b+ia) - Si(a) + \frac{1}{2}\pi\right], \end{aligned} \tag{25}$$

$$(26)\quad \int_0^b \frac{e^{-t}}{a^2+t^2}\,dt = \frac{\sin a}{a}\,[RE_1(b+ia)+Ci(a)] + \frac{\cos a}{a}\left[IE_1(b+ia) - Si(a) + \frac{1}{2}\pi\right],$$

$$(27)\quad \int_0^b \frac{te^{t}}{a^2+t^2}\,dt = \sin a\left[IE_1(-b+ia) - Si(a) + \frac{1}{2}\pi\right] - \cos a\,[RE_1(-b+ia)+Ci(a)],$$

$$(28)\quad \int_0^b \frac{te^{-t}}{a^2+t^2}\,dt = \sin a\left[IE_1(b+ia) - Si(a) + \frac{1}{2}\pi\right] - \cos a\,[RE_1(b+ia)+Ci(a)],$$

$$(29)\quad \int_0^b \frac{\sinh t}{t}\,dt = \frac{1}{2}\,[RE_1(b) - RE_1(-b)],$$

$$(30)\quad \int_0^b \frac{1-\cosh t}{t}\,dt = \log \gamma b + \frac{1}{2}\,[RE_1(b) + RE_1(-b)],$$

$$(31)\quad \int_0^1 \frac{1-\cos at\cosh bt}{t}\,dt = \log\gamma + \frac{1}{2}\log(a^2+b^2) + \tfrac{1}{2}\,[RE_1(b+ia) + RE_1(-b+ia)]$$

where $\log\gamma = 0.5772156649\cdots =$ Euler's constant,

$$(32)\quad \int_0^1 \frac{\sin at\sinh bt}{t}\,dt = \tan^{-1}\frac{b}{a} - \frac{1}{2}\,[IE_1(b+ia) - IE_1(-b+ia)],$$

$$(33)\quad \int_0^1 \frac{\cos at\sinh bt}{t}\,dt = \frac{1}{2}\,[RE_1(b+ia) - RE_1(-b+ia)],$$

$$(34)\quad \int_0^1 \frac{\sin at\cosh bt}{t}\,dt = \frac{1}{2}\,[IE_1(b+ia) + IE_1(-b+ia) + \pi],$$

$$(35)\quad \int_0^b \sin t\log(a^2+t^2)\,dt = \log a^2 - \cos b\log(a^2+b^2) + e^{a}\,[RE_1(a) - RE_1(a+ib)] + e^{-a}\,[RE_1(-a) - RE_1(-a+ib)],$$

$$(36)\quad \int_0^b \cos t\log(a^2+t^2)\,dt = \sin b\log(a^2+b^2) - e^{-a}\,[IE_1(-a+ib)+\pi] - e^{a}\,IE_1(a+ib),$$

$$(37)\quad \int_0^b \sinh t\log(a^2+t^2)\,dt = \cosh b\log(a^2+b^2) - \log a^2 - \sin a\,[IE_1(b+ia) + IE_1(-b+ia) - 2Si(a) + \pi] + \cos a\,[RE_1(b+ia) + RE_1(-b+ia) + 2Ci(a)],$$

$$(38)\quad \int_0^b \cosh t\log(a^2+t^2)\,dt = \sinh b\log(a^2+b^2) + \sin a\,[IE_1(b+ia) - IE_1(-b+ia)] - \cos a\,[RE_1(b+ia) - RE_1(-b+ia)],$$

$$\int_0^b \sin t \tan^{-1}\frac{t}{a}\,dt = -\cos b \tan^{-1}\frac{b}{a} - \frac{e^a}{2} IE_1(a+ib) + \frac{e^{-a}}{2}[\pi + IE_1(-a+ib)], \tag{39}$$

$$\int_0^b \cos t \tan^{-1}\frac{t}{a}\,dt = \sin b \tan^{-1}\frac{b}{a} + \frac{e^a}{2}[RE_1(a+ib) - RE_1(a)] - \frac{e^{-a}}{2}[RE_1(-a+ib) - RE_1(-a)], \tag{40}$$

$$\int_0^b \sinh t \tan^{-1}\frac{t}{a}\,dt = \cosh b \tan^{-1}\frac{b}{a} - \frac{\sin a}{2}[RE_1(b+ia) - RE_1(-b+ia)] - \frac{\cos a}{2}[IE_1(b+ia) - IE_1(-b+ia)], \tag{41}$$

$$\int_0^b \cosh t \tan^{-1}\frac{t}{a}\,dt = \sinh b \tan^{-1}\frac{b}{a} + \sin a\, Ci(a) + \cos a\left[\frac{1}{2}\pi - Si(a)\right] + \frac{\sin a}{2}[RE_1(b+ia) + RE_1(-b+ia)] + \frac{\cos a}{2}[IE_1(b+ia) + IE_1(-b+ia)], \tag{42}$$

$$\int_0^1 \frac{e^{at}\sin bt}{t}\,dt = -\tan^{-1}\frac{b}{a} + IE_1(-a+ib) + \pi, \qquad a > 0, \tag{43}$$

$$\int_0^1 \frac{e^{-at}\sin bt}{t}\,dt = \tan^{-1}\frac{b}{a} + IE_1(a+ib), \qquad a > 0, \tag{44}$$

$$\int_0^1 \frac{e^{-at}(1-\cos bt)}{t}\,dt = \frac{1}{2}\log\left(1+\frac{b^2}{a^2}\right) - RE_1(a) + RE_1(a+ib), \tag{45}$$

$$\int_0^b e^t \log(a^2+t^2)\,dt = e^b \log(a^2+b^2) - \log a^2 - 2\sin a\,[IE_1(-b+ia) - Si(a) + \tfrac{1}{2}\pi] + 2\cos a\,[RE_1(-b+ia) + Ci(a)], \tag{46}$$

$$\int_0^b e^t \tan^{-1}\frac{t}{a}\,dt = e^b \tan^{-1}\frac{b}{a} + \sin a[RE_1(-b+ia) + Ci(a)] + \cos a[IE_1(-b+ia) - Si(a) + \tfrac{1}{2}\pi]. \tag{47}$$

The integrals of equations (19) and (20) have been tabulated by Bleick [4].

7. Extension of the Table of Integrals. Other more complex integrals can be derived from the above table by differentiation or integration with respect to a or b, using the Cauchy-Riemann relations,

$$\frac{\partial RE_1(a+ib)}{\partial a} = \frac{\partial IE_1(a+ib)}{\partial b} = \frac{e^{-a}}{a^2+b^2}[b\sin b - a\cos b] \tag{48}$$

and

$$\frac{\partial RE_1(a+ib)}{\partial b} = -\frac{\partial IE_1(a+ib)}{\partial a}$$

$$= \frac{e^{-a}}{a^2+b^2}[-a\sin b - b\cos b]. \tag{49}$$

Defense Electronic Products Division
Radio Corporation of America
Camden, New Jersey

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Evaluation of the Incomplete Gamma Function of Imaginary Argument by Chebyshev Polynomials

By Richard Barakat

During the course of some work on the diffraction theory of aberrations it was necessary to evaluate numerically the incomplete gamma function of imaginary argument $\gamma(\nu, ix)$ for certain values of the parameter ν. These integrals are special cases of the confluent hypergeometric function, and in the standard notation [1]

$$\gamma(\nu, ix) = \int_0^{ix} e^{-t} t^{\nu-1}\, dt \qquad Re(\nu) > 0$$
$$= (ix)^{\nu} \nu^{-1} {}_1F_1(\nu, 1 + \nu, -ix) \tag{1}$$

where ${}_1F_1$ is the confluent hypergeometric function. Only the case of ν real is considered.

When ν is an integer, ${}_1F_1$ is simply a polynomial in x. It is also possible to evaluate (1) in terms of the Fresnel integrals when $(\nu - 1)$ is a half-integer. For $\nu = \frac{1}{2}$, (1) is proportional to the Fresnel integral. For other half-integer values, we can use the recurrence relation

$$\gamma(1 + \nu, ix) = \nu\gamma(\nu, ix) - (ix)^{\nu} e^{-ix} \tag{2}$$

to generate the necessary formulas. An alternate procedure utilizes the Lommel functions of two variables [2]; unfortunately, the Lommel functions have not been extensively tabulated.

The remaining values of ν can be treated by a Taylor series expansion for small values of x and by asymptotic developments for large values of x. The main difficulty is the intermediate range where x is neither large nor small. A possible method is to use Nielsen's [3] representation

$$\gamma(\nu, ix + iy) - \gamma(\nu, ix) = e^{-ix}(ix)^{\nu-1} \sum_{n=0}^{\infty} (1 - \nu)_n (-ix)^{-n} [1 - e^{-iy} e_n(iy)] \tag{3}$$

where $e_n(iy)$ is the truncated exponential series

$$e_n(iy) = \sum_{m=0}^{n} \frac{(iy)^m}{m!}. \tag{4}$$

This method seems unnecessarily complicated.

A more powerful method is to utilize the Chebyshev polynomials. Instead of expanding the exponential in (1) into a Taylor series, we expand into a series of Chebyshev polynomials. As Lanczos [4] has pointed out, "While the expansion into powers on the basis of the Taylor's series gives the slowest convergence, the expansion into the Chebyshev polynomials gives the fastest convergence."

It is convenient to transform the integral slightly by setting $t = ixq$ in (1)

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$$\gamma(\nu, ix) = (iq)^\nu \int_0^1 e^{-ixq} q^{\nu-1}\, dq. \tag{5}$$

Expanding the exponential into the series [5]

$$e^{-ixq} = \sum_{n=0}^{\infty} \epsilon_n (-i)^n J_n(x) T_n(q) \qquad |q| \leqq 1 \tag{6}$$

where ϵ_n is Newmann's factor ($\epsilon_0 = 2$; $\epsilon_n = 1$ for $n \geqq 1$) and $T_n(q)$ is the Chebyshev polynomial of the first kind

$$\begin{aligned} T_0(q) &= 1 \\ T_n(q) &= \frac{n}{2} \sum_{s=0}^{[n/2]} \frac{(-1)^s (n-s-1)!}{s!(n-2s)!} (2q)^{n-2s}. \end{aligned} \tag{7}$$

Substituting (6) into (5) and interchanging sum and integral (this is permissible because of uniform convergence of the series) leads to the integrals

$$\begin{aligned} B_0(\nu) &\equiv \frac{1}{\nu} \\ B_n(\nu) &\equiv \int_0^1 T_n(q) q^{\nu-1}\, dq \\ &= \frac{n}{2} \sum_{s=0}^{[n/2]} \frac{(-1)^s (n-s-1)!\, 2^{n-2s}}{s!(n-2s)!(n-2s+\nu)}. \end{aligned} \tag{8}$$

Finally (5) becomes

$$\gamma(\nu, ix) = (iq)^\nu \sum_{n=0}^{\infty} \epsilon_n (-i)^n J_n(x) B_n(\nu). \tag{9}$$

To obtain an error estimate on B_n for large n, we use the relation

$$T_n(q) = \cos n\theta, \qquad q = \cos\theta \tag{10}$$

so that (8) becomes

$$\begin{aligned} B_n(\nu) &= \int_0^{\pi/2} \cos n\theta (\cos\theta)^{\nu-1} \sin\theta\, d\theta \\ &\sim 0(1). \end{aligned} \tag{11}$$

The asymptotic expression for $J_n(x)$ when n is large and x is fixed is given by [6]

$$J_n(x) \sim \frac{(x/2)^n}{n!} \qquad \left(\frac{x^2}{4} \ll n\right). \tag{12}$$

Hence the dominating factor is $J_n(x)$.

As an example of the power of the method, we consider the case of the Fresnel integrals ($\nu = \frac{1}{2}$), using as a standard the recent table by Pearcey [7]. The first thirteen values of $B_n(.5)$ are:

$B_0 = 2.0$	
$B_1 = 0.66666667$	$B_7 = -0.49696970$
$B_2 = -1.20$	$B_8 = 0.42976370$
$B_3 = -0.85714286$	$B_9 = 0.40710869$
$B_4 = 0.57777778$	$B_{10} = -0.40752711$
$B_5 = 0.52813853$	$B_{11} = -0.38709818$
$B_6 = -0.54358974$	$B_{12} = 0.35566473$

Omitting the superfluous factor $(iq)^{-.5}$ in (5) yields

$$\gamma_1(.5, ix) = \gamma(.5, ix)(iq)^{-.5}$$

$$= \sum_{n=0}^{\infty} \epsilon_n(-i)^n J_n(x) B_n(.5) \tag{13}$$

$$= \left(\frac{2\pi}{x}\right)^{1/2} [C(x) - iS(x)].$$

The results for various values of x are tabulated below. The numbers in parentheses indicate the number of terms used in the series expansion.

$Re[\gamma_1(.5, ix)]$

	$x = 1$	$x = 2$	$x = 3$	$x = 4$
Pearcey	1.8090490	1.335193	0.811910	0.461462
series	1.80904840 (4)	1.3351931 (5)	0.8119099 (6)	0.4614622 (7)
	1.80902564 (3)	1.3351744 (4)	0.8118993 (5)	0.4614576 (6).

Similar results hold for the imaginary part of γ_1. The small number of terms required for six-place accuracy is remarkable compared with the usual Taylor series expansion.

Using this method, integrals have been computed corresponding to $\nu = .2, .4, .6, .8$ in the interval $0(.1)5$. The integrals in question are

$$\gamma_1(\nu, ix) = \int_0^1 e^{-ixq} q^{\nu-1}\, dq \tag{14}$$

$$= (iq)^{-\nu}\gamma(\nu, ix)$$

(see equation 5).

The first fifteen values of $B_n(\nu)$ are listed in Table 1, while in Table 2 we give the values of the real and imaginary values of $\gamma_1(\nu, ix)$ to six decimals.

TABLE 1

B_n/ν	.2	.4	.6	.8
B_0	5.00000000	2.50000000	1.66666667	1.25000000
B_1	.83333333	.71428571	.62500000	.55555556
B_2	−4.09090909	−1.66666667	−.89743590	−.53571429
B_3	−1.25000000	−.96638655	−.76388889	−.61403509
B_4	3.26839827	.98484848	.32887403	.05952381
B_5	.99358973	.65203859	.42658730	.27324056
B_6	−3.08546292	−.90909091	−.32988751	−.11554623
B_7	−.98290598	−.62150387	−.39839182	−.25726340
B_8	2.86921706	.76839826	.23742520	.05315763
B_9	.90301003	.53065850	.31171679	.18010639
B_{10}	−2.76714376	−.72594072	−.23077198	−.06744618
B_{11}	−.88701757	−.50877478	−.29504438	−.17180411
B_{12}	2.65095170	.65829331	.19072285	.04254734
B_{13}	.8439206	.4620702	.2532754	.1368987
B_{14}	−2.581791	−.630248	−.185156	−.048825
B_{15}	−.624846	−.335736	−.187154	−.108511

TABLE 2

Re[$\gamma_1(\nu, ix)$]					Im[$\gamma_1(\nu, ix)$]				
x/ν	.2	.4	.6	.8	x/ν	.2	.4	.6	.8
0	5.000000	2.500000	1.666667	1.250000	0	0	0	0	0
.1	4.997728	2.497918	1.664744	1.248215	.1	−.083281	−.071380	−.062454	−.055512
.2	4.990925	2.491682	1.658989	1.242871	.2	−.166251	−.142465	−.124630	−.110761
.3	4.979626	2.481327	1.649432	1.233999	.3	−.248598	−.212966	−.186254	−.165486
.4	4.963889	2.466908	1.636128	1.221650	.4	−.330016	−.282593	−.247052	−.219430
.5	4.943798	2.448505	1.619153	1.205896	.5	−.410206	−.351063	−.306759	−.272340
.6	4.919457	2.426217	1.598600	1.186830	.6	−.488874	−.418102	−.365115	−.323971
.7	4.890992	2.400165	1.574586	1.164560	.7	−.565736	−.483443	−.421868	−.374084
.8	4.858551	2.370489	1.547245	1.139217	.8	−.640519	−.546831	−.476778	−.422454
.9	4.822300	2.337349	1.516730	1.110945	.9	−.712964	−.608020	−.529616	−.468863
1.0	4.782427	2.300922	1.483209	1.079908	1.0	−.782825	−.666783	−.580166	−.513108
1.1	4.739134	2.261403	1.446869	1.046282	1.1	−.849872	−.722903	−.628226	−.554999
1.2	4.692643	2.219001	1.407910	1.010259	1.2	−.913890	−.776182	−.673611	−.594363
1.3	4.643186	2.173939	1.366545	.972044	1.3	−.974687	−.826440	−.716151	−.631041
1.4	4.591014	2.126454	1.322998	.931850	1.4	−1.032085	−.873513	−.755697	−.664892
1.5	4.536384	2.076793	1.277506	.889904	1.5	−1.085929	−.917259	−.792115	−.695794
1.6	4.479566	2.025210	1.230312	.846438	1.6	−1.136085	−.957555	−.825293	−.723642
1.7	4.420838	1.971971	1.181667	.801691	1.7	−1.182439	−.994298	−.855137	−.748351
1.8	4.360484	1.917344	1.131828	.755908	1.8	−1.224902	−1.027407	−.881576	−.769857
1.9	4.298791	1.861603	1.081054	.709337	1.9	−1.263406	−1.056824	−.904557	−.788113
2.0	4.236050	1.805022	1.029608	.662227	2.0	−1.297904	−1.082511	−.924050	−.803095
2.1	4.172552	1.747879	.977750	.614827	2.1	−1.328374	−1.104452	−.940044	−.814797
2.2	4.108588	1.690447	.925742	.567385	2.2	−1.354815	−1.122653	−.952550	−.823233
2.3	4.044445	1.632998	.873840	.520143	2.3	−1.377251	−1.137140	−.961600	−.828439
2.4	3.980406	1.575798	.822296	.473342	2.4	−1.395725	−1.147963	−.967244	−.830468
2.5	3.916747	1.519107	.771356	.427214	2.5	−1.410303	−1.155190	−.969553	−.829390
2.6	3.853735	1.463177	.721257	.381983	2.6	−1.421070	−1.158907	−.968616	−.825297
2.7	3.791628	1.408251	.672227	.337863	2.7	−1.428133	−1.159223	−.964540	−.818295
2.8	3.730674	1.354558	.624483	.295059	2.8	−1.431615	−1.156262	−.957449	−.808507
2.9	3.671107	1.302318	.578228	.253763	2.9	−1.431659	−1.150165	−.947483	−.796071
3.0	3.613147	1.251735	.533655	.214153	3.0	−1.428423	−1.141088	−.934797	−.781139
3.1	3.556998	1.203000	.490939	.176394	3.1	−1.422082	−1.129203	−.919558	−.763875
3.2	3.502850	1.156285	.450241	.140635	3.2	−1.412821	−1.114693	−.901946	−.744456
3.3	3.450873	1.111749	.411707	.107010	3.3	−1.400841	−1.097754	−.882152	−.723067
3.4	3.401222	1.069530	.375462	.075636	3.4	−1.386353	−1.078591	−.860375	−.699904
3.5	3.354030	1.029750	.341618	.046611	3.5	−1.369574	−1.057416	−.836824	−.675167
3.6	3.309413	.992512	.310265	.020018	3.6	−1.350732	−1.034451	−.811711	−.649064
3.7	3.267467	.957899	.281477	−.004080	3.7	−1.330060	−1.009920	−.785256	−.621806
3.8	3.228268	.925976	.255308	−.025637	3.8	−1.307793	−.984051	−.757678	−.593607
3.9	3.191871	.896787	.231794	−.044625	3.9	−1.284170	−.957077	−.729200	−.564681
4.0	3.158312	.870358	.210951	−.061033	4.0	−1.259431	−.929226	−.700045	−.535242
4.1	3.127608	.846696	.192780	−.074871	4.1	−1.233815	−.900728	−.670431	−.505501
4.2	3.099754	.825790	.177259	−.086163	4.2	−1.207556	−.871809	−.640578	−.475667
4.3	3.074730	.807608	.164354	−.094951	4.3	−1.180887	−.842691	−.610695	−.445944
4.4	3.052494	.792104	.154010	−.101294	4.4	−1.154033	−.813590	−.580990	−.416526
4.5	3.032988	.779212	.146158	−.105264	4.5	−1.127213	−.784712	−.551660	−.387604
4.6	3.016138	.768853	.140713	−.106951	4.6	−1.100636	−.756258	−.522895	−.359357
4.7	3.001852	.760931	.137576	−.106457	4.7	−1.074503	−.728417	−.494873	−.331955
4.8	2.990025	.755335	.136634	−.103895	4.8	−1.049002	−.701367	−.467764	−.305558
4.9	2.980539	.751945	.137763	−.099392	4.9	−1.024311	−.675274	−.441722	−.280310
5.0	2.973261	.750625	.140826	−.093084	5.0	−1.000593	−.650291	−.416892	−.256347

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Research Laboratory
Itek Corporation
Boston, Massachusetts

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New Tables of Howland's and Related Integrals

By C. W. Nelson

1. Introduction. In an earlier paper by C. B. Ling and the present author [1], values of the four integrals,

$$(1)\qquad \begin{matrix} I_k \\ I_k^* \end{matrix} = \frac{1}{2(k!)} \int_0^\infty \frac{w^k\,dw}{\sinh w \pm w} = \frac{2^k}{k!} \int_0^\infty \frac{x^k\,dx}{\sinh 2x \pm 2x}, \qquad \begin{matrix} (k \geqq 1) \\ (k \geqq 3) \end{matrix}$$

$$(2)\qquad \begin{matrix} II_k \\ II_k^* \end{matrix} = \frac{1}{2(k!)} \int_0^\infty \frac{w^k e^{-w}\,dw}{\sinh w \pm w} = \frac{2^k}{k!} \int_0^\infty \frac{x^k e^{-2x}\,dx}{\sinh 2x \pm 2x}, \qquad \begin{matrix} (k \geqq 1) \\ (k \geqq 3) \end{matrix}$$

were tabulated to 6D. These four integrals are called Howland's integrals because they first appeared in Howland's papers dealing with the stresses in a perforated strip [2, 3]. In another paper, C. B. Ling has reproduced the 6D tables of Howland's integrals from [1] and added tables of other integrals derived from them [4].

2. Values of Howland's Integrals to 9D. For most purposes, 6D tables of Howland's integrals are adequate. This is usually true in plane-stress problems of elasticity involving straight bars of rectangular cross-section, for example. However, the author has found that even in such problems the stresses are more easily evaluated in all regions of interest in the bar without resorting to contour integration if 9D rather than 6D tables of Howland's integrals are available. The 9D values of Howland's integrals given in Table 1 were computed by the author and have been checked by using the following checking formulas [1]:

$$(3)\qquad \begin{cases} \displaystyle\sum_{k=0}^{\infty} (1 - I_{2k+1}) = I_1 - \tfrac{1}{4}, & \displaystyle\sum_{k=1}^{\infty} 2k(1 - I_{2k}) = 2I_2 - \tfrac{1}{8}, \\ \displaystyle\sum_{k=1}^{\infty} (I_{2k+1}^* - 1) = \tfrac{5}{4}, & \displaystyle\sum_{k=2}^{\infty} 2k(I_{2k}^* - 1) = \tfrac{17}{8}, \\ \displaystyle\sum_{k=0}^{\infty} II_{2k+1} = \tfrac{1}{2} - II_1, & \displaystyle\sum_{k=1}^{\infty} 2kII_{2k} = \tfrac{1}{2} - 2II_2, \\ \displaystyle\sum_{k=1}^{\infty} II_{2k+1}^* = \tfrac{1}{2}, & \displaystyle\sum_{k=2}^{\infty} 2kII_{2k}^* = \tfrac{1}{2}. \end{cases}$$

The method of computing the integrals is explained in [1] and this explanation will not be repeated here.

3. Values of I_{2k}, I_{2k}^*, and $\begin{bmatrix} 2 \\ n \end{bmatrix}$ to 18D. Recently the author has encountered a need for still more precise values of some of the Howland's integrals. This occurs in the Hankel transform or Fourier-Bessel integral solutions [5, 6, 7] for problems of elasticity involving axially symmetric loading of a thick plate of infinite radius,

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TABLE 1

Values of Howland's Integrals to 9D

k	I_k	I_k^*	II_k	II_k^*
1	0.768 574 538		0.220 119 581	
2	0.767 847 439		0.087 927 235	
3	0.827 710 296	2.038 710 667	0.043 347 862	0.460 713 719
4	0.883 506 807	1.353 294 115	0.022 583 004	0.099 315 532
5	0.925 475 998	1.156 864 366	0.011 923 473	0.032 412 690
6	0.954 191 562	1.076 729 764	0.006 287 972	0.012 616 908
7	0.972 698 993	1.039 251 312	0.003 295 013	0.005 391 117
8	0.984 124 180	1.020 537 600	0.001 713 298	0.002 433 000
9	0.990 949 179	1.010 870 147	0.000 884 147	0.001 135 997
10	0.994 922 440	1.005 784 842	0.000 453 215	0.000 542 202
11	0.997 188 575	1.003 084 774	0.000 231 010	0.000 262 590
12	0.998 459 958	1.001 644 976	0.000 117 209	0.000 128 431
13	0.999 163 823	1.000 876 180	0.000 059 254	0.000 063 239
14	0.999 549 306	1.000 465 841	0.000 029 870	0.000 031 283
15	0.999 758 559	1.000 247 139	0.000 015 025	0.000 015 525
16	0.999 871 321	1.000 130 809	0.000 007 546	0.000 007 722
17	0.999 931 719	1.000 069 073	0.000 003 785	0.000 003 847
18	0.999 963 903	1.000 036 390	0.000 001 897	0.000 001 919
19	0.999 980 979	1.000 019 128	0.000 000 950	0.000 000 958
20	0.999 990 006	1.000 010 034	0.000 000 476	0.000 000 478
21	0.999 994 762	1.000 005 252	0.000 000 238	0.000 000 239
22	0.999 997 261	1.000 002 744	0.000 000 119	0.000 000 119
23	0.999 998 570	1.000 001 431	0.000 000 060	0.000 000 060
24	0.999 999 255	1.000 000 745	0.000 000 030	0.000 000 030
25	0.999 999 613	1.000 000 388	0.000 000 015	0.000 000 015
26	0.999 999 799	1.000 000 201	0.000 000 007	0.000 000 007
27	0.999 999 896	1.000 000 104	0.000 000 004	0.000 000 004
28	0.999 999 946	1.000 000 054	0.000 000 002	0.000 000 002
29	0.999 999 972	1.000 000 028	0.000 000 001	0.000 000 001
30	0.999 999 986	1.000 000 014	0.000 000 000	0.000 000 000
31	0.999 999 993	1.000 000 007		
32	0.999 999 996	1.000 000 004		
33	0.999 999 998	1.000 000 002		
34	0.999 999 999	1.000 000 001		
35	0.999 999 999	1.000 000 001		
36	1.000 000 000	1.000 000 000		

where integrals such as

$$F(\rho, \alpha) = \int_0^\infty \frac{\left(\sinh \frac{x}{2} + \frac{x}{2} \cosh \frac{x}{2}\right) J_0(\rho x) J_1(\alpha x)\, dx}{\alpha(\sinh x \pm x)} \tag{4}$$

must be evaluated. In equation (4), $J_0(\rho x)$ and $J_1(\alpha x)$ are Bessel functions of order zero and unity, respectively. Sneddon [5, 6] describes an approximate method of evaluating such integrals. He and his associates appear to have treated only cases where the double sign in the denominator has the plus value, but, presumably, their

method could be extended to cases where the double sign takes the minus value. Sneddon's approximate method does not involve the use of Howland's integrals.

For a limiting case of the integral in equation (4), namely $\alpha = 0$ and double sign taken as plus, Sneddon found that his approximate method gave errors of the order of 1% for $\rho = 0$ to $\frac{1}{2}$. However, the author found that Sneddon's approximate method applied to the integral in equation (4) gave an error of about 7% for $\rho = 0$, $\alpha = 1$, and double sign taken plus. Comparable errors in certain other integrals were obtained by the approximate method so that, in a certain thermal stress problem (radius of heated region equal to thickness of plate), the errors in normal stresses on the axis of the plate ranged up to 21.8% of the greatest normal stress occurring anywhere on the axis or 9.3% of the greatest normal stress occurring anywhere in the plate.

The author has felt that it is desirable to be able to obtain the values of integrals such as those considered in the foregoing two paragraphs with an accuracy better than that obtainable by Sneddon's approximate method. The more accurate values may at least be used to check a few values obtained by the approximate method in cases of doubt, and the availability of accurate values of the integrals occurring in thick-plate problems may even be found essential in the extension of the Hankel transform method to problems not yet considered. The author first attempted to evaluate the stresses in a thick-plate problem [7] with the aid of the 9D values of Howland's given in Table 1 and found that only a very limited range of values of the parameters ρ and α in integrals such as that of equation (4) could be successfully dealt with. For this reason, the 18D values of two of the four Howland's integrals, I_k and I_k^*, for even integral k, were computed. These are given in Table 2 and discussed further in the following. Green and Willmore [9] encountered integrals similar to those discussed in the present paper except that theirs contained only one Bessel function. They used Howland's integrals for the simpler integrals they had to evaluate and Sneddon's approximate method for the rest. Apparently the Fourier-Bessel integral method was first applied to thick plate problems by Lamb [10], who made no attempt to evaluate the integrals. Dougall [11] proposed that integrals such as the example in equation (4), but containing only one Bessel function, be evaluated by contour integration, but apparently no one has cared to perform the task involved which appears to be considerable even though the roots of $\sinh z \pm z = 0$ are known.

If the integral taken as an example in equation (4) is expressed as a series of Howland integrals by expanding the numerator of the integrand in powers of x, only even powers of x will occur in this expansion. Thus the integral can be evaluated if adequate values of I_k or I_k^* are available for even integral k, and if the series converges or can somehow be summed. From a consideration of various loading conditions of a thick plate, it appears to the author that as long as the axially-symmetric loading of the plate consists only of boundary loads on the plane faces (i.e. no body forces), all other integrals needed can also be evaluated from tables of I_k and I_k^* for even integral k. Accordingly, the values of I_k and I_k^* given to 18D in Table 2 are believed to be the only Howland integrals needed in a fairly broad class of thick-plate problems. Again the reader is referred to [1] for the general method of computing the integrals in Table 2. However, it should be stated here that the method depends on the basic equation

TABLE 2

Values of the Howland Integrals I_{2k} and I^*_{2k} to 18D

k	I_k	I_k^*
2	0.767 847 439 133 919 047	
4	0.883 506 806 508 692 590	1.353 294 115 170 484 009
6	0.954 191 561 826 139 064	1.076 729 763 674 217 127
8	0.984 124 180 148 424 614	1.020 537 600 064 959 340
10	0.994 922 439 853 445 374	1.005 784 842 224 523 505
12	0.998 459 957 947 832 383	1.001 644 976 253 520 390
14	0.999 549 305 562 626 177	1.000 465 841 012 174 418
16	0.999 871 321 426 371 778	1.000 130 808 809 455 335
18	0.999 963 903 069 165 323	1.000 036 389 774 380 024
20	0.999 990 005 851 207 354	1.000 010 033 628 030 387
22	0.999 997 260 778 826 414	1.000 002 744 455 935 026
24	0.999 999 255 282 148 058	1.000 000 745 402 213 094
26	0.999 999 798 878 365 743	1.000 000 201 210 025 403
28	0.999 999 945 988 927 609	1.000 000 054 022 369 865
30	0.999 999 985 565 214 614	1.000 000 014 436 216 216
32	0.999 999 996 158 384 196	1.000 000 003 841 795 571
34	0.999 999 998 981 377 142	1.000 000 001 018 645 284
36	0.999 999 999 730 790 958	1.000 000 000 269 211 822
38	0.999 999 999 929 059 585	1.000 000 000 070 940 758
40	0.999 999 999 981 355 380	1.000 000 000 018 644 662
42	0.999 999 999 995 111 469	1.000 000 000 004 888 537
44	0.999 999 999 998 721 023	1.000 000 000 001 278 977
46	0.999 999 999 999 666 045	1.000 000 000 000 333 955
48	0.999 999 999 999 912 959	1.000 000 000 000 087 041
50	0.999 999 999 999 977 351	1.000 000 000 000 022 649
52	0.999 999 999 999 994 116	1.000 000 000 000 005 884
54	0.999 999 999 999 998 473	1.000 000 000 000 001 527
56	0.999 999 999 999 999 604	1.000 000 000 000 000 396
58	0.999 999 999 999 999 898	1.000 000 000 000 000 102
60	0.999 999 999 999 999 974	1.000 000 000 000 000 026
62	0.999 999 999 999 999 993	1.000 000 000 000 000 007
64	0.999 999 999 999 999 998	1.000 000 000 000 000 002
66	1.000 000 000 000 000 000	1.000 000 000 000 000 000

$$\begin{matrix} I_k \\ I_k^* \end{matrix} = \sum_{n=1}^{\infty} (\mp 1)^{n+1} \begin{bmatrix} k \\ n \end{bmatrix} \tag{5}$$

where

$$\begin{bmatrix} k \\ n \end{bmatrix} = \frac{1}{2(k!)} \int_0^{\infty} \frac{x^{n+k-1}\, dx}{\sinh^n x} \tag{6}$$

and that the values in Table 2 have been checked by using the second and fourth of equations (3). When these checks were first applied, errors were indicated. To eliminate these errors, they had to be located and this was facilitated by applying another checking formula†

† For $n = 1$, use

$$\sum_{m=1}^{\infty} \left(\begin{bmatrix} 2m \\ 1 \end{bmatrix} - 1 \right) = \frac{3}{4} - \ln 2 \tag{7a}$$

$$(7)\qquad \sum_{m=1}^{\infty} \frac{2m}{2m+1}\left[\begin{matrix}2m\\ n\end{matrix}\right] = \frac{1}{2(n-1)}, \qquad n = 2, 3, 4, \cdots$$

Equation (7) is obtained by writing the series of integrals represented by the left member of the equation, interchanging the order of summation and integration, performing the summation which requires merely recognition of the Maclaurin expansion of $\cosh x - \dfrac{\sinh x}{x}$ and then performing the evaluation of the resulting elementary integral as follows:

$$(8)\qquad \frac{1}{2}\int_0^{\infty} \frac{x^{n-2}(x\cosh x - \sinh x)\,dx}{\sinh^n x} = -\frac{x^{n-1}}{2(n-1)\sinh^{n-1} x}\Bigg|_0^{\infty} = \frac{1}{2(n-1)}.$$

TABLE 3

Values of $\left[\begin{matrix}2\\ n\end{matrix}\right]$ *to* 18D

n	$\left[\begin{matrix}2\\ n\end{matrix}\right]$	n	$\left[\begin{matrix}2\\ n\end{matrix}\right]$
1	1.051 799 790 264 645 000	21	0.036 396 239 976 639 766
2	0.450 771 338 684 847 857	22	0.034 712 213 240 700 944
3	0.283 656 164 817 032 302	23	0.033 177 094 523 417 174
4	0.206 411 435 020 280 449	24	0.031 771 973 852 978 737
5	0.162 091 502 769 567 468	25	0.030 481 012 454 106 163
6	0.133 389 997 047 394 398	26	0.029 290 843 691 027 776
7	0.113 302 756 992 243 931	27	0.028 190 108 942 707 765
8	0.098 463 515 074 533 508	28	0.027 169 094 240 899 480
9	0.087 055 830 293 630 464	29	0.026 219 443 043 962 235
10	0.078 014 012 373 164 903	30	0.025 333 927 165 280 539
11	0.070 671 880 431 666 952	31	0.024 506 262 569 448 302
12	0.064 591 720 775 032 384	32	0.023 730 960 108 696 094
13	0.059 474 138 718 298 180	33	0.023 003 203 705 225 661
14	0.055 107 451 486 079 519	34	0.022 318 750 267 406 497
15	0.051 337 774 971 088 869	35	0.021 673 846 947 047 716
16	0.048 050 565 748 631 057	36	0.021 065 162 331 082 628
17	0.045 158 809 284 589 149	37	0.020 489 728 904 903 199
18	0.042 595 221 342 283 185	38	0.019 944 894 690 604 315
19	0.040 306 957 521 379 715	39	0.019 428 282 397 575 541
20	0.038 251 937 970 666 201	40	0.018 937 754 758 470 797

TABLE 4

Values of $\displaystyle\int_0^{\infty} \frac{xJ_1(\rho x)\,dx}{\sinh x + x}$

ρ	$\int_0^{\infty} \frac{xJ_1(\rho x)\,dx}{\sinh x + x}$
0	0
1	0.471 282
2	0.250 403
3	0.166 664
4	0.125 000
5	0.100 000

The values of $\begin{bmatrix} 2 \\ n \end{bmatrix}$ are recorded in Table 3, since they are the starting point for obtaining I_k and I_k^*, and because they may be needed as in the example leading to Table 4 in the following. It was not felt necessary to include values of $\begin{bmatrix} m \\ n \end{bmatrix}$ for $m = 4, 6, 8, \cdots$ because, at worst, only a few such values might be needed by a reader, and these can be computed from Table 3 by using equation (16) given in the following.

4. Application of Howland's Integrals in Evaluating Related Integrals. Procedures for evaluating integrals such as those in equation (4) for moderate values of ρ and α, say for ρ and α each less than unity, have been amply discussed in connection with similar integrals treated in references [2], [3], [4] and [7]. If either ρ or α (or both) should be considerably greater than unity in integrals such as those of equation (4), some special study of the integral in question is usually required and an effort should be made to determine some simple expression which the value of the integral approaches asymptotically for large values of the parameter or parameters. Usually this can be done by considering the physical problem in which the integral arose and by examining the known approximate solution for a limiting case of the problem. For example, suppose the integral $\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x + x}$ is to be evaluated for a series of values of ρ covering the range $0 \leqq \rho < \infty$. The integral arises in a three-dimensional elasticity problem involving axially symmetric loads on a thick plate. By considering a limiting case of the physical problem or by other methods, it can be shown that

$$\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x + x} \approx \frac{1}{2\rho}, \qquad \text{for large } \rho. \tag{9}$$

Thus the range of values of ρ which must be considered is reduced to the range from zero to the lowest value at which equation (9) gives the result with sufficient accuracy. The integral in question may be expressed as a series of integrals in the form

$$\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x + x} = \sum_{n=1}^{\infty} (-1)^{n+1} \int_0^\infty \frac{x^n J_1(\rho x)\,dx}{\sinh^n x} \tag{10}$$

where the integrals in the right member may be evaluated by contour integration following the method described for a similar integral in [8]. For example, the first two integrals required are

$$\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x} = \frac{1}{\rho} - 2\pi[\kappa_1(\pi\rho) - 2\kappa_1(2\pi\rho) + 3\kappa_1(3\pi\rho) - \cdots] \tag{11}$$

$$\int_0^\infty \frac{x^2 J_1(\rho x)\,dx}{\sinh^2 x} = \frac{1}{\rho} + 2\pi[\kappa_1(\pi\rho) + 2\kappa_1(2\pi\rho) + 3\kappa_1(3\pi\rho) + \cdots] - 2\pi^2\rho[\kappa_0(\pi\rho) + 4\kappa_0(2\pi\rho) + 9\kappa_0(3\pi\rho) + \cdots] \tag{12}$$

where $\kappa_0(x)$ and $\kappa_1(x)$ are modified Bessel functions of the second kind of order zero and one respectively.

By evaluating the first four integrals and applying an Euler transformation to

the resulting alternating series in equation (10), the values of the left member of equation (10) for $\rho = 4$ and $\rho = 5$ were found to be as given in Table 4. The entries in Table 4 were completed as follows. For $\rho = 0$ and $\rho = 1$, the integral was evaluated from

$$\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x + x} = 2\rho I_2 - 2 \cdot \frac{3}{2}\rho^3 I_4 + 2 \cdot \frac{3 \cdot 5}{2 \cdot 4}\rho^5 I_6 - \cdots \tag{13}$$

For $\rho = 2$ and $\rho = 3$, a Kummer transformation was applied to equation (13) so as to obtain

$$\int_0^\infty \frac{xJ_1(\rho x)\,dx}{\sinh x + x} = \sum_{n=1}^{4} (-1)^{n+1} \int_0^\infty \frac{x^n J_1(\rho x)\,dx}{\sinh^n x} + \text{Remainder Series} \tag{14}$$

where

$$\begin{aligned}\text{Remainder Series} = {} & 2\rho\left\{I_2 - \sum_{n=1}^{4} (-1)^{n+1}\begin{bmatrix}2\\ n\end{bmatrix}\right\} \\ & - 2 \cdot \tfrac{3}{2}\rho^3\left\{I_4 - \sum_{n=1}^{4} (-1)^{n+1}\begin{bmatrix}4\\ n\end{bmatrix}\right\} + \cdots\end{aligned} \tag{15}$$

The remainder series converges for $\rho < 5$ but the convergence is, of course, very slow as ρ approaches 5 and, even for $\rho = 2$ and $\rho = 3$, it is best to speed the convergence by applying an Euler transformation. To evaluate the series in equation (15), it is necessary to use Table 2, Table 3, and the equation

$$\begin{aligned}\begin{bmatrix}m\\ n\end{bmatrix} = \frac{1}{(m-1)m}\Bigg\{ & \frac{(n+m-2)(n+m-1)}{n^2}\begin{bmatrix}m-2\\ n\end{bmatrix} \\ & - \frac{(n+1)}{n}\begin{bmatrix}m-2\\ n+2\end{bmatrix}\Bigg\}\end{aligned} \tag{16}$$

which is equivalent to the first of equations (4) in reference [1].

5. Acknowledgments. Except for Table 4, the tables presented in this paper were computed while the author was employed as Associate Professor of Mechanics and Design, University of California, Berkeley, California, prior to June 30, 1959.

Applied Physics Laboratory
GPD Development Laboratory
International Business Machines Corporation
Endicott, New York

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Two Theorem Tables of Matrix Algebra

By Gilbert C. Best

Abstract. It is shown in this paper how it is possible, by a slight modification of the mode of presentation, to state many of the theorems of matrix algebra in table form. Due to the large number of theorems presented the tables give only the statement of each theorem. It is not considered feasible to also attempt to indicate sources and proofs.

1. Introduction. Many theorems are of such a nature that they can be stated in table form. Take, for example, the theorem which reads, "If A is a non-singular symmetric matrix then so is A^{-1}." As a first step we restate this in "operator" form. We define the "inverse-forming" operator O^{-1} to be such that $O^{-1}A = A^{-1}$, where A is assumed non-singular. Then the theorem reads, "The property of symmetry is invariant for the operator O^{-1}." Finally, defining "v" to indicate invariance, the theorem may be presented in an "operator vs. property" table by a v in row O^{-1} and column "symmetry." This theorem appears in Table I in row 3 column 6. This method of presentation has the obvious advantage of considerable compactness.

Again, consider the theorem

$$(A^T)^{-1} = (A^{-1})^T. \tag{1}$$

Defining a "transposing" operator O^T such that $O^TA = A^T$ this theorem can be rewritten in "operator" form as :

$$O^TO^{-1}A = O^{-1}O^TA. \tag{2}$$

The theorem can then be stated as: "The operators O^T and O^{-1} are commutative." In this last form the theorem in question is readily presented in an "operator vs. operator" table by defining k to indicate commutativeness. Then (1) is given as a k in row O^T and column O^{-1}, or, equally well, by a k in row O^{-1} and column O^T.

2. Operators. In general, any transformation of a matrix may be considered to define an operator. Also, for implied given fixed coefficients $\alpha_1, \cdots \alpha_m$ and an ordered set $(A_1, A_2, \cdots, A_m)$ of matrices, the linear combination $\sum_{i=1}^{m} \alpha_i A_i$ may be considered an operator. Similarly, any function of a matrix or matrices may be considered an operator. For example, the forming of the determinant, the extraction of the characteristic roots or characteristic vectors, or the product of two matrices, may all be considered operators. A numbered list of the operators used in this paper is shown below. The operators in the rows of Table 1 and the rows and columns of Table 2 preserve the same numbering.

3. Properties. The idea of a property also may be generalized beyond the well-known and clearly defined properties such as symmetry, skew-symmetry, singularity, positive definiteness, etc. For example, the fact that a given matrix A has specific roots, e.g., 2, -5, 7 say, is a property, and this property is unaffected by a

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List of Operators

#	Symbol	Explanation
0	I	Identity Operator
1	O^T	Transposer
2	$\bar{O}$	Conjugator, i.e., sends $A = (a_{ij})$ into $\bar{A} = (\bar{a}_{ij})$
3	O^{-1}	Inverse Former (if inverse exists)
4	O^A	Adjoint Former (or adjugate, see [1])
5	$O^{(k)}$	kth Compound Former, see [1]
6	PQ	Equivalent Matrix Former*
7	PP^T	Congruent Matrix Former*
8	$P\bar{P}^T$	Hermitely Congruent Matrix Former*
9	PP^{-1}	Similar Matrix Former*
10	$PP^\perp$	Orthogonally Similar Matrix Former*
11	$UU^\perp$	Unitarily Similar Matrix Former*
12	$\sum$	Sum Former
13	O^α	Former of Product with Scalar $\alpha \neq 0$
14	$LNCO$	Former of Linear Combination (coefficients assumed fixed)*
15	$\prod$	Former of Product of Matrices
16	O^P	Former of Matrix Raised to Power P
17	POL	Former of a Polynomial from a Matrix (coefficients assumed fixed)*
18	$FUNC$	Function Taker*
19	DET	Determinant Taker
20	tr	Trace Taker
21	R	Characteristic Root Taker
22	V	Characteristic Vector Taker

* Operator matrices or coefficients are over the same field as the matrix A to which they are applied unless otherwise indicated.

List of Properties

#	*Property*
1	Diagonal
2	Block Diagonal (also called "direct sum")
3	Triangular. 2 types* : upper and lower
4	Block Triangular. 2 types* : upper and lower
5	Non-Singular
6	Symmetric
7	Hermitian
8	Skew Symmetric
9	Skew Hermitian
10	Orthogonal
11	Unitary
12	Normal
13	Diagonable, i.e., similar to a diagonal matrix
14	Symmetric Positive Definite (real)
15	Hermitian Positive Definite
16	Definite. 4 types* positive definite, positive semi-definite, negative semi-definite, negative definite
17	Specific Rank
18	Specific Index
19	Specific Invariant Factors
20	Specific Characteristic Equation or Characteristic Roots
21	Specific Characteristic Vectors
22	Specific Determinant

* v in Table 1 implies that the matrix remains the same type.

similarity transformation. If a matrix has a specific determinant, rank, or index, that is also a property. A list of properties used is shown below, and the columns of Table 1 preserve the same numbering.

4. Rules of Application. If an operator is applied to more than one matrix at a time the following rules hold:

(1) If the operator O is applicable to a single matrix, i.e., is a matrix-to-matrix operator, and O is applied to the ordered set $(A_1, A_2, \cdots, A_m)$, then one obtains $(OA_1, OA_2, \cdots, OA_m)$.

(2) If the operator O is applicable to two matrices and yields one matrix and O is applied to the ordered set $(A_1, A_2, \cdots, A_m)$, then O is first applied to A_1, A_2 yielding A_2', then O is applied to $A_2'A_3$ yielding A_3', etc.

If a set of matrices $(A_1, A_2, \cdots, A_m)$ is said to have a property P, then each matrix of the set has property P.

TABLE 1

Operators vs. Properties

	Op.	Diagonal 1	Block Diagonal 2	Triangular 3	Block Triangular 4	Non-singular 5	Symmetric 6	Hermitian 7	Skew Symmetric 8	Skew Hermitian 9	Orthogonal 10	Unitary 11	Normal 12	Diagonable 13	Sym. Pos. Def. 14	Herm. Pos. Def. 15	Definite 16	Rank (Has Specific) 17	Index 18	Invariant Factors 19	Ch. Roots 20	Ch. Vectors 21	Determinant 22	
1	O^T	v	v	t	t	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v		v	1
2	$\bar{O}$	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v					2
3	O^{-1}	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v			v		3
4	O^A	v	v	v	v	v	v	v			v	v	v	v	v	v	a					v		4
5	$O^{(k)}$	v		v		v	v	v				v												5
6	PQ					v												v		v				6
7	$P\bar{P}^T$					v	v		v						v			v	v	v				7
8	$P\bar{P}^T$					v		v		v						v	v	v		v				8
9	PP^{-1}					v								v				v		v	v		v	9
10	$PP^{\perp}$					v	v		v		v			v	v			v	v	v	v		v	10
11	$UU^{\perp}$					v		v		v		v	v			v	v	v		v	v		v	11
12	$\sum$	v	v	v	v		v	v	v	v					v	v	b							12
13	O^{α}	v	v	v	v	v	v	r	v	r			v	v	$+$	$+$	$+$	v	$+$	v		v		13
14	$LNCO$	F	F	F	F		F	v	v	v														14
15	$\prod$	v	v	v	v	v	k	k			v	v			k	k								15
16	O^P	v	v	v	v	v	v	v			v	v	v	v	v	v						v		16
17	POL	F	F	F	F		F	v					F		r	r						v		17
18	$FUNC$	F	F	F	F		F	v					F		r	r						v		18
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	

Symbol	Explanation
a	maintains type for even n but for n odd: neg→pos
b	maintains type, though a change from semi into non-semi is possible
F	invariant for $LNCO$, POL, $FUNC$ over a field F for matrices over F
k	invariant if and only if the matrices commute
r	invariant for α, $LNCO$, POL, $FUNC$ over the real field
t	changes type
v	invariant
$+$	for $\alpha > 0$

TABLE 2
Operators vs. Operators

	Op.	O^T	$\bar{O}$	O^{-1}	O^A	$O^{(k)}$	PQ	PP^T	$P\bar{P}^T$	PP^{-1}	$PP^{\perp}$	$UU^{\perp}$	$\sum$	O^n	$LNCO$	$\prod$	O^P	POL	$FUNC$	DET	tr	R	V	
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	
1	O^T	I	k	k	k	k		k			k		k	k	k	o	k	k	k	k	k	k		1
2	$\bar{O}$	k	I	k	k	k							k	k	r	k	k	r	r	k	k	k	k	2
3	O^{-1}	k	k	I	k	k				k	k	k				o	k			k		k		3
4	O^A	k	k	k						k	k	k				o	k							4
5	$O^{(k)}$	k	k	k												k	k							5
6	PQ												k	k	k									6
7	$P\bar{P}^T$	k											k	k	k									7
8	$P\bar{P}^T$												k	k	k									8
9	PP^{-1}			k	k								k	k	k	k	k	k	k	k	k			9
10	$PP^{\perp}$	k		k	k								k	k	k	k	k	k	k	k	k			10
11	$UU^{\perp}$			k	k								k	k	k	k	k	k	k	k	k			11
12	$\sum$	k	k				k	k	k	k	k	k		k							k			12
13	O^n	k	k				k	k	k	k	k	k	k		k						k	k		13
14	$LNCO$	k	r				k	k	k	k	k	k		k							k	k		14
15	$\prod$	o	k	o	o	k				k	k	k								k				15
16	O^P	k	k	k	k	k				k	k	k								k		k		16
17	POL	k	r							k	k	k										k		17
18	$FUNC$	k	r							k	k	k										k		18
19	DET	k	k	k						k	k	k				k	k							19
20	tr	k	k							k	k	k	k	k	k									20
21	R	k	k	k										k	k		k	k	k					21
22	V		k																					22
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	

Symbol	Explanation
I	two applications of operator equivalent to the identity operator
o	order of product reverses, e.g. if P = operator for row in table $P\prod(AB) = \prod P(BA) = P(B)*P(A)$
k	operators are commutative
r	operators are commutative for *LNCO*, *POL*, *FUNC* over the real field

3703 Tulsa Way
Fort Worth 7, Texas

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Improved Formulas for Complete and Partial Summation of Certain Series

By Herbert E. Salzer and Genevieve M. Kimbro

Abstract. In two previous articles one of the authors gave formulas, with numerous examples, for summing a series either to infinity (complete) or up to a certain number n of terms (partial) by considering the sum of the first j terms S_j, or some suitable modification S_j', closely related to S_j, as a polynomial in $1/j$. Either S_∞ or S_n was found by m-point Lagrangian extrapolation from S_{j_0}, S_{j_0-1}, $\cdots$, S_{j_0-m+1} to $1/j = 0$ or $1/j = 1/n$ respectively. This present paper furnishes more accurate m-point formulas for sums (or sequences) S_j which behave as even functions of $1/j$. Tables of Lagrangian extrapolation coefficients in the variable $1/j^2$ are given for: complete summation, $m = 2(1)7$, $j_0 = 10$, exactly and 20D, $m = 11$, $j_0 = 20$, 30D; partial summation, $m = 7$, $j_0 = 10$, $n = 11(1)25(5)100, 200, 500, 1000$, exactly. Applications are made to calculating π or the semi-perimeters of many-sided regular polygons, Euler's constant,

$$1 + \sum_{r=1}^{j} \left\{ \frac{-1}{(4r-1)^2} + \frac{1}{(4r+1)^2} \right\} = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots \quad \text{for} \quad j = \infty$$

(Catalan's constant), calculation of a definite integral as the limit of a suitably chosen sequence, determining later zeros of $J_\nu(x)$ from earlier zeros for suitable ν, etc. A useful device in many cases involving sums of odd functions, is to replace S_j by a trapezoidal-type S_j' which is seen, from the Euler-Maclaurin formula, to be formally a series in $1/j^2$. In almost every example, comparison with the earlier $(1/j)$-extrapolation showed these present formulas, for 7 points, to improve results by anywhere from around 4 to 9 places.

1. Introduction. In two earlier papers, [1, 2], one of the authors gave tables for both complete summation (all terms, to infinity) and partial summation (up to a certain number of terms) of certain kinds of slowly convergent series. In the case of partial summation, divergent series were also included, provided that a suitable auxiliary series could be found of the desired slowly convergent type and simply related to the original divergent series. The essential idea in both cases is to regard the sequence S_j, the sum of the first j terms of the series, as the values for $x = 1/j$ of an interpolable function $S(x)$ to which the slight extrapolation from specified S_j, to $j = \infty$ $(x = 0)$ or to $j = k$ $(x = 1/k)$, $k > j_0$ where S_{j_0} is the last specified S_j, yields good accuracy. The approximating formula for $S(x)$ was an m-point Lagrange polynomial of the $(m-1)$th degree in x which at $x = 1/j$ assumes the prescribed value S_j, for the last m values of j ending at $j_0 = 5$, 10, 15 or 20, from which we extrapolated to either $j = \infty$ $(x = 0)$ or $j = k > j_0 (x = 1/k)$. Numerous examples which yielded surprisingly high accuracy for a variety of sequences S_j in both complete and incomplete cases, attested to the wide applicability of considering S_j a smooth function of $1/j$, even when we were in complete ignorance as to the

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actual analytic expression for $S_j = S(1/j)$ or of a theoretical justification for considering $S(1/j)$ as an approximate polynomial in $1/j$.

However, a still further improvement in m-point formulas for both complete and partial summation is applicable to a wide class of sequences where $S_j \equiv S(1/j)$ behaves like an even function of $1/j$. Thus by taking x^2 as argument instead of x, in an m-point Lagrangian extrapolation formula for $x = 0$ (complete summation) or a value close to 0 (partial summation) based upon those same final m values of S_j, we should get accuracy equivalent to $(2m - 1)$th degree instead of $(m - 1)$th degree. As will be seen from the illustrations below, the resulting improvement is often quite impressive.

There is no hard and fast classification of all the varied problems to which these newer formulas are applicable. The reason is that even if a problem does not seem offhand to involve a sequence of that even-function type, often with a very slight transformation, regrouping, or alteration, one sees that it really is amenable to this more accurate treatment.

Of course, every sequence to which these improved extrapolation formulas for arguments $1/j^2$ are particularly applicable can also be handled by the earlier formulas employing arguments $1/j$, because any polynomial in x^2 is also a polynomial in x, but with considerably less accuracy for the same number m of points and the same last $j = j_0$. But the converse is not true—we cannot in general expect these newer summation formulas to work well when applied indiscriminately to sequences where the earlier method may give very high accuracy. One way of realizing this is to think of the non-constant part of a well-behaved function of x near $x = 0$ being approximated by Cx. Extrapolation employing $x^2 = y$ as the variable, near $x = 0$, is like extrapolation for $\sqrt{y}$ based upon a polynomial approximation in the variable y. But, as anybody who has attempted to interpolate in a table of square roots near zero has found out, $\sqrt{y}$, although continuous at $y = 0$, possesses a singularity due to an infinite derivative.

2. Other Related Articles. The idea of the extrapolation to $x = 0$ for argument $y = x^2$ has been employed for just the linear case in the well-known "h^2-extrapolation process", or "deferred approach to the limit", which has been extensively treated in the literature on the numerical solution of differential equations (first introduced by L. F. Richardson [3, 4]). The argument x or h corresponds to two conveniently small values of a mesh-length, say h_1 and h_2. Richardson's process has been generalized to higher powers beyond h^2 by several writers, notably G. Blanch, [5] and H. C. Bolton and H. I. Scoins [6]. However, the only reference that was encountered by the writer which was concerned with problems where the approximation might be considered as a purely even function of h having more than a single term, has been M. G. Salvadori [7]. Besides some sets of 2-point coefficients for h^2- and h^4-extrapolation, Salvadori tabulates 3-point coefficients for (h^2, h^4)- and (h^4, h^6)- extrapolation, and 4-point coefficients for (h^2, h^4, h^6)- and (h^4, h^6, h^8)-extrapolation. The values of h are in the form $1/n_i$, where n_i are sets of small integers ranging from 2 to 8. Salvadori gives applications to numerical differentiation and integration, as well as to some boundary value problems and characteristic value problems.

3. Formulas for Complete Summation. In choosing a j_0 suitable for most complete summation purposes, we wish to obtain a substantial increase in accuracy over the use of the earlier formulas in [1], which has already been proved to be very accurate, without having coefficients that might be too cumbersome. It is also desirable to give exact values rather than decimal values, because in highly accurate formulas the theoretical or truncation error might be considerably smaller than the computing error arising from the use of rounded decimal entries. But we must also take account of the fact that the fixed points $1/j^2$ in place of the older $1/j$ makes the exact fractional form of the extrapolation coefficients have around twice as many digits in both numerator and denominator, which adds considerably to the amount of time to do an example.

In the present paper it seems that a very convenient choice is $j_0 = 10$, for all cases ranging from the 2-point through the 7-point. In other words we give formulas for linear through sextic Lagrangian extrapolation formulas for functions of the variable $y = x^2$ taken at $x = 1/j$, or arguments $y = 1/j^2$ at $j = 10, 9, \cdots, 10 - m + 1$ for $m = 2(1)7$. This is equivalent to quadratic through twelfth degree accuracy for even functions in $x = 1/j$. The extrapolation formula to obtain the complete sum S from the partial sums S_{10}, S_9, $\cdots$, S_{10-m+1} is the very simple

$$S \sim \sum_{i=0}^{m-1} A_{10,10-i}^{(m)} S_{10-i} . \tag{1}$$

The coefficients $A_{10,10-i}^{(m)}$ are given in Table 1 in exact fractional form $B_{10,10-i}^{(m)}/D_{10}^{(m)}$, so that (1) may be most conveniently employed as

$$S \sim (1/D_{10}^{(m)}) \sum_{i=0}^{m-1} B_{10,10-i}^{(m)} S_{10-i} . \tag{2}$$

In no case through $m = 7$, does $D_{10}^{(m)}$ have more than ten digits exclusive of final zeros, which is convenient in the division. The values of $A_{10,10-i}^{(m)}$ are given also to 20 decimals in Table 2.

Although the 7-point formulas for $j_0 = 10$ are very accurate, as will be apparent from the examples below, we give also in Table 3 for possible use in some kind of isolated key calculation where extreme accuracy is sought, even at the expense of considerable computing labor, the coefficients in the 11-point formula, ending at S_{20}, given exactly, to be employed in

$$S \sim (1/D_{20}^{(11)}) \sum_{i=0}^{10} B_{20,20-i}^{(20)} S_{20-i} . \tag{3}$$

Formula (3) is exact for any even polynomial in $x = 1/j$ up to the 20th degree. To avoid too much non-essential numerical work, no illustrations were given of the use of Table 3, since the resulting accuracy is so high by comparison with the results of using Table 1 or 2, that an excessively large number of significant digits is needed to reveal its full extent. But Table 3 should be kept in reserve for a summation problem requiring unusual precision.

The formula for $A_{j_0,j_0-i}^{(m)}$ is obtained rather simply from the well-known definition of the m-point Lagrangian interpolation coefficients where we have fixed points $1/j_0^2, 1/(j_0 - 1)^2, \cdots, 1/(j_0 - m + 1)^2$ and set the variable $y = x^2 = 1/j^2$ equal to 0 to correspond to $j = \infty$.

TABLE 1

$$A^{(m)}_{10,10-i} = B^{(m)}_{10,10-i}/D^{(m)}_{10}$$

$m = 2$

$B^{(2)}_{10,9}$	$= -81$
$B^{(2)}_{10,10}$	$= 100$
$D^{(2)}_{10}$	$= 19$

$m = 3$

$B^{(3)}_{10,8}$	$= 19456$
$B^{(3)}_{10,9}$	$= -59049$
$B^{(3)}_{10,10}$	$= 42500$
$D^{(3)}_{10}$	$= 2907$

$m = 4$

$B^{(4)}_{10,7}$	$= -67\ 05993$
$B^{(4)}_{10,8}$	$= 398\ 45888$
$B^{(4)}_{10,9}$	$= -717\ 44535$
$B^{(4)}_{10,10}$	$= 400\ 00000$
$D^{(4)}_{10}$	$= 13\ 95360$

$m = 5$

$B^{(5)}_{10,6}$	$= 2034\ 43488$
$B^{(5)}_{10,7}$	$= -23001\ 55599$
$B^{(5)}_{10,8}$	$= 82879\ 44704$
$B^{(5)}_{10,9}$	$= -1\ 17517\ 54833$
$B^{(5)}_{10,10}$	$= 56875\ 00000$
$D^{(5)}_{10}$	$= 1269\ 77760$

$m = 6$

$B^{(6)}_{10,5}$	$= -75703\ 12500$
$B^{(6)}_{10,6}$	$= 17\ 57751\ 73632$
$B^{(6)}_{10,7}$	$= -123\ 97838\ 67861$
$B^{(6)}_{10,8}$	$= 359\ 05926\ 59456$
$B^{(6)}_{10,9}$	$= -448\ 74915\ 24087$
$B^{(6)}_{10,10}$	$= 200\ 20000\ 00000$
$D^{(6)}_{10}$	$= 3\ 35221\ 28640$

$m = 7$

$B^{(7)}_{10,4}$	$= 54190\ 40768$
$B^{(7)}_{10,5}$	$= -31\ 54296\ 87500$
$B^{(7)}_{10,6}$	$= 474\ 59296\ 88064$
$B^{(7)}_{10,7}$	$= -2761\ 33679\ 65995$
$B^{(7)}_{10,8}$	$= 7181\ 18531\ 89120$
$B^{(7)}_{10,9}$	$= -8388\ 15723\ 34857$
$B^{(7)}_{10,10}$	$= 3575\ 00000\ 00000$
$D^{(7)}_{10}$	$= 50\ 28319\ 29600$

$$(4) \qquad A^{(m)}_{j_0,j_0-i} = \frac{(-1)^{m-1}(j_0 - i)^{2m-2}}{\prod_{k=0}^{m-1}{}' \left[(j_0 - k)^2 - (j_0 - i)^2\right]},$$

where in $\prod'$, $k = i$ is omitted.

4. Illustrations of Complete Summation.

A. *Example* 1. Considering the circle as the limiting case of inscribed regular polygons of j sides, as $j \to \infty$, the quantity π is the limit of the semi-perimeter, $j \sin \alpha$, where $\alpha = 180°/j = \pi/j$, as $j \to \infty$.* Now the approximation $S_j = j \sin \alpha =$

* Although this example affords a splendid illustration of the improvement of $(1/j^2)$-extrapolation over $(1/j)$-extrapolation, it suffers from the aesthetic defect of having the value of π occurring implicitly in every S_j in the various powers of α needed to compute $\sin \alpha$. In other words, there is definitely something "circular" in this example.

TABLE 2

$A_{10,10-i}^{(m)}$ *in Decimal Form*

$m = 2$

$A_{10,9}^{(2)} = -4.26315\ 78947\ 36842\ 10526$
$A_{10,10}^{(2)} = 5.26315\ 78947\ 36842\ 10526$

$m = 3$

$A_{10,8}^{(2)} = 6.69281\ 04575\ 16339\ 86928$
$A_{10,9}^{(2)} = -20.31269\ 34984\ 52012\ 38390$
$A_{10,10}^{(2)} = 14.61988\ 30409\ 35672\ 51462$

$m = 4$

$A_{10,7}^{(4)} = -4.80592\ 32026\ 14379\ 08497$
$A_{10,8}^{(4)} = 28.55599\ 12854\ 03050\ 10893$
$A_{10,9}^{(4)} = -51.41650\ 54179\ 56656\ 34675$
$A_{10,10}^{(4)} = 28.66643\ 73351\ 67985\ 32278$

$m = 5$

$A_{10,6}^{(5)} = 1.60219\ 78021\ 97802\ 19780$
$A_{10,7}^{(5)} = -18.11463\ 36098\ 54198\ 08949$
$A_{10,8}^{(5)} = 65.27083\ 72237\ 78400\ 24899$
$A_{10,9}^{(5)} = -92.54970\ 97523\ 21981\ 42415$
$A_{10,10}^{(5)} = 44.79130\ 83361\ 99977\ 06685$

$m = 6$

$A_{10,5}^{(6)} = -0.22583\ 03039\ 55303\ 95530$
$A_{10,6}^{(6)} = 5.24355\ 64435\ 56443\ 55644$
$A_{10,7}^{(6)} = -36.98404\ 36201\ 18987\ 76605$
$A_{10,8}^{(6)} = 107.11111\ 74954\ 31220\ 92142$
$A_{10,9}^{(6)} = -133.86654\ 44631\ 80008\ 84564$
$A_{10,10}^{(6)} = 59.72174\ 44482\ 66636\ 08913$

$m = 7$

$A_{10,4}^{(7)} = 0.01077\ 70418\ 88152\ 99926$
$A_{10,5}^{(7)} = -0.62730\ 63998\ 75844\ 32029$
$A_{10,6}^{(7)} = 9.43840\ 15984\ 01598\ 40160$
$A_{10,7}^{(7)} = -54.91570\ 11329\ 03951\ 53140$
$A_{10,8}^{(7)} = 142.81482\ 33272\ 41627\ 89522$
$A_{10,9}^{(7)} = -166.81830\ 92541\ 16626\ 40765$
$A_{10,10}^{(7)} = 71.09731\ 48193\ 65042\ 96325$

TABLE 3

$A_{20,20-i}^{(11)} = B_{20,20-i}^{(11)}/D_{20}^{(11)}$

$B_{20,10}^{(11)} = 74096\ 20000\ 00000\ 00000\ 00000$
$B_{20,11}^{(11)} = -35\ 37615\ 48335\ 31708\ 54782\ 90644$
$B_{20,12}^{(11)} = 649\ 45974\ 08685\ 61313\ 24915\ 22048$
$B_{20,13}^{(11)} = -6200\ 60319\ 26092\ 91850\ 74192\ 35023$
$B_{20,14}^{(11)} = 34801\ 60376\ 52150\ 35629\ 23772\ 47744$
$B_{20,15}^{(11)} = -1\ 21941\ 46052\ 37160\ 60638\ 42773\ 43750$
$B_{20,16}^{(11)} = 2\ 73659\ 70208\ 28851\ 47761\ 53823\ 64160$
$B_{20,17}^{(11)} = -3\ 92511\ 27655\ 98026\ 11495\ 97941\ 97770$
$B_{20,18}^{(11)} = 3\ 47343\ 22454\ 05086\ 94470\ 03616\ 05120$
$B_{20,19}^{(11)} = -1\ 72481\ 59320\ 99496\ 29170\ 21217\ 51885$
$B_{20,20}^{(11)} = 36718\ 51008\ 00000\ 00000\ 00000\ 00000$
$D_{20}^{(11)} = (32124\ 40751)(38698\ 35264)(23\ 58125)$
$= 2\ 93153\ 05663\ 14310\ 15219\ 20000$

$S(1/j)$ is seen to be an even function of $1/j$ which equals π for $1/j = 0$. Therefore we expect an m-point Lagrange polynomial approximation for variable $1/j^2$ to be considerably more accurate than a polynomial in $1/j$. Following are the values of the semi-perimeters $j \sin \alpha$ to 25D, which were obtained from a table of $\sin \alpha$ to 30D originally published by Herrmann [8]. For $j = 4(1)6, 9, 10$, $\sin \alpha$ was copied

from Herrmann's table, and for $j = 7, 8$, $\sin \alpha$ was computed by Taylor's theorem employing Herrmann's entries as key values:

j	S_j: Semi-perimeter $= j \sin \alpha$
4	2.82842 71247 46190 09760 33774
5	2.93892 62614 62365 64584 35298
6	3.00000 00000 00000 00000 00000
7	3.03718 61738 22906 84333 03783
8	3.06146 74589 20718 17382 76799
9	3.07818 12899 31018 59739 68965
10	3.09016 99437 49474 24102 29342

In the above values of S_j, as well as S_j given in the other examples, the accuracy of the last few places, although highly probable, is still not absolutely guaranteed. However, in every example the values of S_j are certainly correct up to the number of places needed to guarantee that the "computational error" in the final answer (which is due to the error in the S_j multiplied by the extrapolation coefficients $A_{10,j}^{(\infty)}$) is appreciably less than the deviation of the answer from the true value. This latter "truncating error" is thus made to stand out clearly, and it indicates the theoretical accuracy of the extrapolation formula, regardless of the number of places carried in the work. In practice we do not often know at the outset of an example just how many places are needed in the S_j to assure us that the computing error will be dominated by the truncating error. Sometimes when the theoretical accuracy turns out to be unexpectedly fine, the example must be done again, carrying more places, to prevent the computing error from obscuring the truncating error.

The results of the extrapolations employing (1) or (2), for $m = 7$, gave for π, (whose true value to 20D is 3.14159 26535 89793 23846), the answer 3.14159 26535 89793 179 ... which is correct to within a unit in the 16th decimal. The extent of the improvement over the earlier $(1/j)$-extrapolation formulas is apparent from the result of 3.14159 280 ... obtained by the corresponding 7-point $(1/j)$-extrapolation coefficients, which deviates from π by $1\frac{1}{2}$ units in the 7th decimal. In other words, the error in the use of this newer formula is only around $0.4 \cdot 10^{-9}$ of that in the older one. The greater power of this newer method in this present example may be further illustrated even for $m = 4$, where $(1/j^2)$-extrapolation yields 3.14159 2650 ..., or accuracy to around $\frac{1}{2}$ of a unit in the 8th decimal, whereas the corresponding 4-point $(1/j)$-extrapolation formula gives no better than 3.1411 ..., which is off by $\frac{1}{2}$ of a unit in the 3rd decimal. In fact, the answer even by 2-point $(1/j^2)$-extrapolation, namely 3.1413 ..., is still better than the above 3.1411....

It is interesting to note that the use of $(1/j^2)$-extrapolation on the semi-perimeters gives this great improvement only for the *inscribed* polygons, and it will not work well for the *circumscribed* polygons, upon which it was also tried. A reason that would lead us to expect poor extrapolation results, even though the corresponding semi-perimeter $j \tan \alpha$ is still an even function of $1/j$, is that the series for $\tan \alpha$ converges poorly by comparison with $\sin \alpha$. Thus for $\alpha = \pi/4$, occurring in $S_j = S_4$, the remainder after the term involving the sixth power of $1/j^2$, is con-

siderably greater for $j \tan \alpha$, so that the use of (1) or (2) for $m = 7$ is not nearly so good as for $j \sin \alpha$.

B. *Example* 2. The sequence for Euler's constant

$$\gamma = \lim_{j\to\infty}\left\{\sum_{r=1}^{j}(1/r) - \log j\right\} = 0.57721\ 56649\ 01532\ 86061 \text{ to 20D}$$

has been treated earlier by $(1/j)$-extrapolation ([1], p. 358). Applying (1) or (2), for seven points, directly to $S_j = \sum_{r=1}^{j}(1/r) - \log j$ yields the very inaccurate 0.593, the reason being that S_j does not behave like an even function of $1/j$. The older $(1/j)$-extrapolation formulas, employing $j_0 = 10$, gave 0.57721 41 ... and 0.57721 56695 ... by the 4- and 7-point formulas with respective errors of around $1\frac{1}{2}\cdot 10^{-6}$ and $\frac{1}{2}\cdot 10^{-8}$. To improve upon these results we must modify our S_j sequence into an even function of $1/j$ having the same limit γ. This is easily accomplished by replacing the last $1/r$ in the summation, namely $1/r = 1/j$, by half its value, or $1/2j$. At first sight there is an apparent motivation in that the new finite summation is suggestive (at one end anyhow) of the more accurate trapezoidal rather than rectangular approximation to the integral $\int_1^j (1/r)\,dr$. This trapezoidal motivation happens to lead to the correct choice in this present example, but in general it does not yield a sequence that is even in $(1/j)$. The true motivation lies in the Euler-Maclaurin summation formula applied to $\log j$. The general formula is expressible as

$$\frac{1}{w}\int_a^{a+jw} f(x)dx = \left(\frac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{j-1} + \frac{1}{2}f_j\right) - \frac{w}{12}(f_j' - f_0') + \frac{w^3}{720}(f_j''' - f_0''') - \frac{w^5}{30240}(f_j^v - f_0^v) + \cdots \text{[9]}. \tag{5}$$

Now (5) does not denote a complete equality, since the Euler-Maclaurin formula is an asymptotic expression that is given with a remainder term. Employing (5) heuristically for $w = 1$, $a = 1$ and $f(x) = 1/x$, the right member of (5), exclusive of the $(\frac{1}{2}f_0 + f_1 + \cdots + f_{j-1} + \frac{1}{2}f_j)$ and an undisclosed remainder term, is an even function of $1/(j+1)$, from which, replacing j by $j-1$,

$$\int_1^j (1/x)dx - \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j-1} + \frac{1}{2j}\right)$$

is an even function of $1/j$, so that the same is true of the sequence

$$S_j' \equiv \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j-1} + \frac{1}{2j}\right) - \log j$$

whose limit, as $j \to \infty$, is also equal to γ.*

Since the older m-point $(1/j)$-extrapolation formula is linear in S_j (or S_j') and

* The reader is cautioned that the above heuristic demonstration is not to be understood as a proof that we have a convergent infinite series in $(1/j^2)$ from which we can "prove" that the "constant" term in S_j' is γ by taking the limit as $j \to \infty$. The fallacy there would be in that there is no "constant" term because the $f_0, f_0', f_0''', \cdots$ terms in (5) yield for $f(x) = 1/x$ a divergent sequence. Actually S_j' is defined only up to any fixed order derivative, say $f_{j-1}^{\{2\}}$, and it then consists of terms in $1/j^2$, constant terms and an integral formula for the remainder.

yields exactly zero for any polynomial in $1/j$ having no constant term, up to the $(m-1)$th degree, the above-mentioned 4- and 7-point results will not be changed by use of S_j' instead of S_j. But the improvement is very noticeable when S_j' is employed with $(1/j^2)$-extrapolation. Following are the terms in the modified sequence S_j' to 20D:

j	$S_j' = \sum_{r=1}^{j-1} \frac{1}{r} + \frac{1}{2j} - \log j$
4	0.57203 89722 13442 71450
5	0.57389 54208 99232 95873
6	0.57490 71974 38611 66585
7	0.57551 84223 73258 12347
8	0.57591 56011 77306 92889
9	0.57618 81210 76479 02991
10	0.57638 31609 74208 28424

The use of the 7-point formula in (1) or (2), where $j_0 = 10$, upon S_j', gave an answer of 0.57721 56649 0143 ... which is correct to a unit in the 13th decimal (i.e., 5 places more than $(1/j)$-extrapolation). Use of just the 4-point formula in (1) or (2) gave an answer as good as 0.57721 56647 5 ... which is correct to within $1\frac{1}{2}$ units in the 10th decimal (i.e., 4 places more than $(1/j)$-extrapolation).

C. *Example* 3. A different type of sequence is encountered in the evaluation of the definite integral $\int_0^1 \frac{1}{1+x}\,dx = \log 2$, whose value to 20D is 0.69314 71805 59945 30942. One obvious sequence to consider is S_j which is formed by dividing the interval (0, 1) into j equally spaced intervals and letting S_j be the sum of the rectangles of height $1/[1 + (r-1)/j]$ and width $1/j$, for $r = 1(1)j$, but that fails to behave as an even function of $1/j$. However, the trapezoidal rule, or

$$S_j' = \frac{1}{j}\left(\frac{1}{2} + \frac{1}{1+1/j} + \frac{1}{1+2/j} + \cdots + \frac{1}{1+(j-1)/j} + \frac{1}{4}\right),$$

according to the Euler-Maclaurin formula (5), where now $w = 1/j$, $a = 0$, and both $f_j^{(p)}$ is fixed as well as $f_0^{(p)}$, being at the endpoints 1 and 0, is seen to have a truncating error that is formally a series in $1/j^2$. The values of S_j', in either exact form, or to 20D, are as follows:

j	$S_j' = \frac{1}{j}\left(\frac{1}{2} + \sum_{r=1}^{j-1} \frac{1}{1+r/j} + \frac{1}{4}\right)$
4	1171/1680 = 0.69702 38095 23809 52381
5	1753/2520 = 0.69563 49206 34920 63492
6	9631/13860 = 0.69487 73448 77344 87734
7	2 50241/3 60360 = 0.69441 94694 19469 41947
8	2 00107/2 88288 = 0.69412 18503 71850 37185
9	5 66803/8 16816 = 0.69391 76020 05837 29995
10	1615 04821/2327 92560 = 0.69377 14031 75427 94323

The 4- and 7-point $(1/j)$-extrapolation, $j_0 = 10$, gave values of 0.69314 86 ... and 0.69314 7176 ..., correct to $1\frac{1}{2}$ units in the 6th decimal and $\frac{1}{2}$ unit in the 8th

decimal respectively. The $(1/j^2)$-extrapolation was performed for every m-point formula from $m = 2$ through $m = 7$, with the following results:

m	value of S	deviation	m	value of S	deviation
2	0.69314 81...	10^{-6}	5	0.69314 71805 67...	10^{-11}
3	0.69314 7188...	10^{-8}	6	0.69314 71805 6054...	$\frac{1}{2}\cdot 10^{-12}$
4	0.69314 71807 1...	$1\frac{1}{2}\cdot 10^{-10}$	7	0.69314 71805 60046...	10^{-13}

The improvement over $(1/j)$-extrapolation in the 4- and 7-point results is by four and five places respectively.

D. *Example* 4. A somewhat more sophisticated application of $(1/j^2)$-extrapolation is in the summation of the series for Catalan's constant, or

$$T_2 = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots.$$

H. T. Davis [10] gives a full discussion of Catalan's constant, including an account of the earlier work of J. W. L. Glaisher, and he also reprints Glaisher's 32-decimal value of $T_2 = 0.91596\ 55941\ 77219\ 01505\ 46035\ 14932\ 38$. Since the series for T_2 is absolutely convergent, it may be regrouped as

$$T_2 = 1 + \left(-\frac{1}{3^2} + \frac{1}{5^2}\right) + \left(-\frac{1}{7^2} + \frac{1}{9^2}\right) + \cdots + \left(-\frac{1}{(4r-1)^2} + \frac{1}{(4r+1)^2}\right) + \cdots.$$

The general term u_r, $r > 0$, of T_2 is equal to $\dfrac{-16r}{(16r^2-1)^2}$, which is an odd function of r or $1/r$. Thus, as in the preceding example, employing (5) with $w = a = 1$, the modified sum

$$S_j' = S_j - \tfrac{1}{2}u_j = 1 - \sum_{r=1}^{j-1} \frac{16r}{(16r^2-1)^2} - \frac{1}{2}\left(\frac{16j}{(16j^2-1)^2}\right)$$

is again seen to be formally an even function of $1/j$, having the same limit S which is approached by S_j.* The values of S_j' to 20D are as follows:

	$S_j' = 1 - \sum_{r=1}^{j-1} \frac{16r}{(16r^2-1)^2} - \frac{1}{2}\left(\frac{16j}{(16j^2-1)^2}\right)$
4	0.91798 69831 73330 85103
5	0.91724 36100 54163 02747
6	0.91684 71757 66868 06945
7	0.91661 06554 47552 03321
8	0.91645 81601 71966 79489
9	0.91635 40724 61230 31205
10	0.91627 98501 91732 37910

* Although in Example 4 we know the explicit formula for $\int_1^j f(x)\,dx$, we may expect this principle to be applicable also in cases where $\int_1^j f(x)\,dx$, $f(x)$ odd, or for that matter also $f_j^{(p)}$ for odd p, is not known in closed form, and where S_j' may still be regarded formally as a series in $1/j^2$.

Use of the 7-point $(1/j)$-extrapolation, $j_0 = 10$, upon either S_j or S_j', while not identical in accuracy, because now the difference of $\frac{1}{2}(16j/(16j^2 - 1)^2)$ is no longer an exact polynomial in $1/j$, gave results very close to each other, namely 0.91596 55973 ... and 0.91596 55980 ... with respective deviations of $\frac{1}{3}\cdot 10^{-8}$ and $\frac{2}{3}\cdot 10^{-8}$. The use of $(1/j^2)$-extrapolation, i.e., (1) or (2), for $m = 7$, while giving the poorer answer of 0.91596 74 ... with a deviation of $2\cdot 10^{-6}$ in working with the S_j sequence (as was to be expected), gave upon working with the S_j' sequence the highly accurate 0.91596 55941 7714 ..., which is correct to $\frac{4}{5}\cdot 10^{-12}$, showing a gain in accuracy of around 5 places.

5. Formulas for Partial Summation. Given the first ten terms of a sequence S_j which behaves as an even function of $1/j$, we might wish to find by $(1/j^2)$-extrapolation S_n, $n > 10$, instead of going to the limit as $j \to \infty$. The purpose of this section is to improve what was accomplished in [2] where just $(1/j)$-extrapolation was employed. The m-point formula for S_n which occurs usually as a sum of the form $\sum_{r=0}^{n} u_r$, is obtained by setting $x = 1/n^2$ in the Lagrange interpolation coefficients whose fixed points are $1/j_0^2$, $1/(j_0 - 1)^2$, $\cdots$, $1/(j_0 - m + 1)^2$. In the present instance, in order to avoid too much tabulation, since now besides j_0 and m, n is also a variable, being no longer just ∞, we consider a choice of j_0 and m which shall be suitable for most problems and which shall give a substantial increase in accuracy over the $(1/j)$-extrapolation formulas previously given which were based upon $j_0 = 10$ and $m = 7$ [2]. Thus it is natural to take $j_0 = 10$ and $m = 7$ for these present formulas also. The argument $n = 11(1)25(5)100, 200, 500, 1000$, and all coefficients are given exactly. This range of n is not quite so extensive as in the previous paper because the arguments $1/j^2$ in place of $1/j$, $j = 4, 5, \cdots, 10$, n, increase the labor in computing the exact forms, which also have considerably greater bulk in figures. To find $S_n \equiv S(n)$, we employ the extrapolation formula in the following form:

$$S(n) = \sum_{j=4}^{10} A_j(n) S_j . \tag{6}$$

Every set of coefficients $A_j(n)$ is given in the exact fractional form of $C_j(n)/D(n)$ where $D(n)$ is the least common denominator for each n. Thus it may help the computer to have

$$S(n) = (1/D(n)) \sum_{j=4}^{10} C_j(n) S_j . \tag{7}$$

In (6) and (7) the $j_0 = 10$ is understood as well as $m = 7$. When also n is understood, we may employ for (7) the somewhat more concise

$$S_n = (1/D) \sum_{j=4}^{10} C_j S_j . \tag{7'}$$

In (7), or (7'), the $D(n)$, or D, is given also in the form of factors having no more than 10 digits, exclusive of terminal 0's, to facilitate the divisions on a ten-bank desk calculator. The $C_j(n)$ and $D(n)$ are shown in Table 4.

TABLE 4 $A_j(n) \equiv C_j(n)/D(n)$

TABLE 4 $A_j(n) = C_j(n)/D(n)$

n	$C_6(n)$	$-C_5(n)$	$C_4(n)$	n
11	4 33523 26144	276 00097 65625	4690 09522 11456	11
12	2963 53792	1 85546 87500	30 76065 53856	12
13	41 05720 29952	2539 20898 43750	41364 52401 80736	13
14	3 11594 84416	190 91796 87500	3070 02326 69664	14
15	73 47078 43072	4469 00781 25000	71153 79028 62336	15
16	299 91003 75040	18137 20703 12500	2 86535 50491 68640	16
17	98713 24662 98880	59 41749 02343 75000	932 85993 94858 59840	17
18	202 53664 87040	12144 04296 87500	1 89697 11668 10720	18
19	5 14830 54912 30720	307 69771 72851 56250	4786 28907 41312 96256	19
20	33 51761 38752	1997 80546 87500	30967 19121 46176	20
21	2980 34491 55584	1 77232 05566 40625	27 39045 93898 29120	21
22	2557 65176 64768	1 51794 04296 87500	23 39958 38615 45424	22
23	11486 99740 73344	6 80571 09375 00000	104 68292 53359 69792	23
24	517 62000 03584	30621 58203 12500	4 70115 09625 81248	24
25	18277 34070 23104	10 79839 46875 00000	165 50605 49407 21152	25
30	47 47239 09632	2791 67890 62500	42538 14131 63616	30
35	19 94826 94088 78592	1169 85066 73242 18750	17764 31856 19235 88096	35
40	55614 22677 27872	32 55672 19140 62500	493 29102 44672 84736	40
45	1 10518 08503 88992	64 61947 71523 43750	977 63645 22693 67296	45
50	314 67009 69601 92512	18382 81470 45703 12500	2 77821 11083 92383 05056	50
55	12260 77786 28307 51744	7 15812 21397 61250 00000	108 09688 47762 36090 65472	55
60	2 10143 48218 04032	122 62754 30078 12500	1850 73872 20503 06816	60
65	1 99146 82341 59643 68896	116 16710 49246 78906 25000	1752 43060 92065 47783 18848	65
70	2 63102 15141 94944	153 42833 85078 12500	2313 69106 74489 04572	70
75	27053 53085 29138 40128	15 77253 55481 87207 03125	237 77936 68838 83951 47264	75
80	1138 17615 39236 16768	66344 10184 61484 37500	9 99933 39136 72116 19584	80
85	1 98436 64585 46096 70144	115 64971 16095 43261 71875	1742 71873 87812 89262 36672	85
90	21 91251 31803 36128	1276 89772 02734 37500	19238 30729 92601 52264	90
95	17 01791 62058 22944 21504	991 56366 90583 06093 75000	14937 24378 14420 81022 44352	95
100	27624 63417 03513 53856	16 09416 13444 06640 62500	242 41878 50975 44215 71328	100
200	17 09077 65240 57655 37792	995 03809 78214 71484 37500	14975 38393 17401 02043 49696	200
500	82 10561 52187 30411 82835 34336	4779 34787 72793 93251 99609 37500	71912 84793 65397 78913 65856 13568	500
1000	186 00631 75612 24045 59830 53028 72064	10827 09010 57365 44610 56489 80078 12500	1 62905 31822 30989 18580 33518 15589 49632	1000

TABLE 4—*Continued*

n	$-C_7(n)$	$C_8(n)$	$-C_9(n)$	n
11	32215 59596 03275	1 05827 99417 34400	1 76151 30190 31997	11
12	203 46692 18547	628 35371 54048	932 01747 03873	12
13	2 66745 13455 15117	7 92802 85920 78848	11 04949 07557 46163	13
14	19442 06520 05475	56307 02125 05600	75493 41510 13713	14
15	4 44575 22425 25195	12 63888 61612 85120	16 50602 94005 59083	15
16	17 71857 77781 80125	49 67921 56241 92000	63 66611 34021 56463	16
17	5721 68313 61299 25965	15871 91324 13418 53696	20054 82572 16792 85845	17
18	11 55895 58305 65507	31 79469 79994 82880	39 73673 74457 90385	18
19	29008 53338 24768 97375	79250 25550 94548 48000	98190 51035 15985 81345	19
20	1 86850 45656 98995	5 07620 03723 05920	6 24536 43402 04353	20
21	164 65175 07237 67125	445 23348 97725 44000	544 62440 88208 18755	21
22	140 21515 98577 29411	377 66930 76995 60448	459 75489 79673 51217	22
23	625 57532 85960 23526	1679 36327 39394 84672	2036 05740 52839 83961	23
24	28 02756 84854 84925	75 02445 18166 52800	90 64378 27268 21583	24
25	984 70190 61233 70365	2629 30613 59658 59840	3167 20040 82195 30606	25
30	2 51281 64849 05545	6 65213 12473 90720	7 93146 86729 95923	30
35	1 04501 17314 93605 18895	2 75278 96205 65685 04320	3 26324 47322 86967 02113	35
40	2894 18470 98839 01945	7600 17480 22670 13120	8976 92198 97040 61283	40
45	5725 63039 44139 07145	15004 08135 79219 76320	17679 14149 53778 25463	45
50	16 25027 50478 80584 46845	42 52100 27375 85566 51520	50 01673 53498 68138 89743	50
55	631 69035 64270 86020 26140	1651 10814 77140 03986 02240	1939 75352 82211 59219 40191	55
60	10807 62370 15787 49045	28225 74314 90409 26720	33129 03631 38264 76023	60
65	10227 95435 17712 52380 23885	26694 90068 27877 68605 08160	31309 54142 10812 86737 65769	65
70	13497 88556 74941 83765	35211 73936 27440 74240	41275 01812 78067 39991	70
75	1386 70343 48980 87560 96555	3616 01493 44620 31514 82880	4236 72540 71523 70324 98717	75
80	58 29847 70830 98366 73455	151 97106 54462 40406 73280	177 99104 37154 06907 98677	80
85	10158 07546 77808 62362 83515	26472 62121 05369 52358 50240	30995 56816 28717 48010 98541	85
90	1 12115 49286 84852 73055	2 92113 94047 04369 86880	3 41934 10964 90406 39717	90
95	87035 63171 86878 79918 38615	2 26725 41556 61183 16566 11840	2 65335 47692 58504 18153 45331	95
100	1412 31425 76716 92152 80985	3678 43631 10728 17912 21760	4304 05053 46689 91848 03459	100
200	87160 02410 86950 16496 14895	2 26755 16422 07280 30737 20320	2 64979 65188 21093 77455 55413	200
500	4 18434 18289 57096 52184 08777 48535	10 88253 25190 80068 46170 98172 82560	12 71247 00401 14305 40998 13477 44429	500
1000	9 47848 63514 29742 74436 93853 54534 22965	24 65030 27747 54660 71683 78425 93408 61440	28 79387 20044 35172 31507 54843 52414 47671	1000

TABLE 4—*Continued*

1000	9 47848 63514 29742 74436 93853 54534 22965	24 65030 27747 54660 71683 78425 93408 61440	28 79387 20044 35172 31507 54843 52414 47671	1000

TABLE 4—*Continued*

n	$C_{10}(n)$	$D(n)$	n
11	1 43000 00000 00000	44879 52578 71103 = (1 90333) (23579 47691)	11
12	568 75000 00000	90 55414 51776 = (1 31072) (69 08733)	12
13	6 00600 00000 00000	60575 02131 84506 = (7 42586) (8157 30721)	13
14	38542 96875 00000	2796 73094 52992 = (14144) (19773 26743)	14
15	8 10409 60000 00000	45878 30507 81250 = (19 53125) (2348 96922)	15
16	30 43906 25000 00000	1 42056 90230 86592 = (1 65376) (85899 34592)	16
17	9406 54000 00000 00000	376 37396 52504 25606 = (539 54566) (69757 57441)	17
18	18 37214 84375 00000	64870 92642 37248 = (1 86048) (34867 84401)	18
19	44894 85000 00000 00000	1429 80143 77167 40006 = (15995 59934) (8938 71739)	19
20	2 83032 75000 00000	8268 80000 00000 = 82688 × 10^5	20
21	245 05000 00000 00000	6 65527 25103 02199 = (33 65793) (19773 26743)	21
22	205 64098 24218 75000	5 24745 22458 77512 = (22 25432) (23579 47691)	22
23	906 19200 00000 00000	21 91462 44320 20321 = (64 36343) (34048 25447)	23
24	40 17406 25000 00000	92727 44466 18624 = (69 08733) (1342 17728)	24
25	1398 70016 00000 00000	30 99441 52832 03125 = (50 78125) (61035 15625)	25
30	3 46064 46875 00000	6643 01250 00000 = 664 30125 × 10^6	30
35	1 41427 09838 40000 00000	2494 83022 65195 31250 = (12617 18750) (19773 26743)	35
40	3874 39102 05000 00000	64 74956 80000 00000 = 647 49568 × 10^9	40
45	7609 15064 00000 00000	122 58226 40976 56250 = (351 56250) (34867 84401)	45
50	21 48569 72361 28125 00000	33721 92382 81250 00000 = (5525) (61035 15625) × 10^6	50
55	832 08552 92288 00000 00000	12 81116 27096 61894 53125 = (19 53125) (30 50969) (2143 58881)	55
60	14196 11500 45000 00000	215 41075 20000 00000 = 2154 10752 × 10^9	60
65	13405 46811 80201 60000 00000	201 12800 04714 18007 81250 = (19 53125) (126 23962) (8157 30721)	65
70	17660 88429 01562 50000	262 61370 80546 87500 = (1328 12500) (19773 26743)	70
75	1811 88546 04312 00000 00000	26 74891 94869 99511 71875 = (3 90625) (11 21931) (61035 15625)	75
80	76 08787 21924 75000 00000	1 11669 14969 60000 00000 = (13) (85899 34592) × 10^9	80
85	13245 41781 61905 60000 00000	193 44878 97051 94082 03125 = (19 53125) (241 37569) (4103 38673)	85
90	1 46077 09919 92812 50000	2124 75924 43593 75000 = (6093 75000) (34867 84401)	90
95	1 13325 68971 63248 00000 00000	1642 69466 64944 16367 18750 = (39 06250) (470 45881) (8938 71739)	95
100	1837 89011 11730 25000 00000	26 56250 00000 00000 00000 = 2 65625 × 10^{16}	100
200	1 12987 07515 96336 75000 00000	1600 00000 00000 00000 00000 = 16 × 10^{22}	200
500	5 41841 72385 59042 30937 75000 00000	7629 39453 12500 00000 00000 00000 = (2048) (61035 15625)^2 × 10^6	500
1000	12 27206 94867 61018 36245 33412 25000 00000	17265 62500 00000 00000 00000 00000 00000 = 172 65625 × 10^{27}	1000

The coefficients $A_j(n) \equiv C_j(n)/D(n)$ were calculated directly from the formula

$$A_j(n) = \frac{j^{12}}{n^{12}} \frac{\prod_{k=4}^{10}{}' (n^2 - k^2)}{\prod_{k=4}^{10}{}' (j^2 - k^2)}, \tag{8}$$

where $k = j$ is absent from $\prod'$. Both the calculation of $A_j(n)$ and the determination of $D(n)$ was facilitated by expressing each of the factors in the right member of (8) in terms of powers of primes.

To facilitate the use of (8) for desired values of n other than in this present table, we notice that we may express $A_j(n)$ as

$$A_j(n) = B_j \cdot \frac{\prod_{k=4}^{10}{}' (n^2 - k^2)}{n^{12}}, \quad \text{where} \tag{9}$$

$$B_j = \frac{j^{12}}{\prod_{k=4}^{10}{}' (j^2 - k^2)} \tag{10}$$

is independent of n. The exact, as well as 30 decimal, values of the fundamental quantities B_j are given in the following Schedule 1.

SCHEDULE 1

j	$B_j = j^{12} \Big/ \prod_{k=4}^{10}{}' (j^2 - k^2)$
4	$\frac{65536}{60\ 81075} =$ 0.01077 70418 88152 99926 41103 75221
5	$-\frac{97\ 65625}{155\ 67552} =$ −0.62730 63998 75844 32028 87647 33209
6	$\frac{2\ 36196}{25025} =$ 9.43840 15984 01598 40159 84015 98402
7	$-\frac{1\ 38412\ 87201}{2520\ 46080} =$ −54.91570 11329 03951 53140 25117 94669
8	$\frac{2684\ 35456}{18\ 79605} =$ 142.81482 33272 41627 89522 26664 64497
9	$-\frac{3\ 13810\ 59609}{1881\ 15200} =$ −166.81830 92541 16626 40764 80794 74705
10	$\frac{390\ 62500}{5\ 49423} =$ 71.09731 48193 65042 96325 41775 64463

6. Illustrations of Partial Summation.

A. *Example* 5. Suppose that in Example 1 above, instead of passing to the limit as $j \to \infty$ to obtain π, we wished to calculate S_{20}, or the semi-perimeter of a 20-sided regular polygon from the semi-perimeters of the 4- through 10-sided regular polygons. We have $S_{20} = 20 \sin 9°$, whose value to 20D is 3.12868 93008 04617

38020. Using the same values of S_j as in Example 1, we find by the earlier method of $(1/j)$-extrapolation [2] $S_{20} = 3.12868\ 93076\ \ldots$ which is correct to around a unit in the 8th decimal. But use of the present tables for $(1/j^2)$-extrapolation in (6) or (7), for $n = 20$, yields the highly accurate $S_{20} = 3.12868\ 93008\ 04617\ 359\ \ldots$, correct to about 2 units in the 17th decimal, showing a gain of around 9 places.

B. *Example* 6. As an illustration of a different type of problem that does not correspond to one in complete summation, consider the case where from the first few known zeros of some higher mathematical function, we wish to obtain the value of some later zero, say the nth. As will be seen below, there are circumstances when it is preferable to choose as the sequence S_j, $j \leqq j_0$, from which to extrapolate, some suitable even function of $1/j$ which may not be a function of the jth root, and yet from S_j, $j > j_0$, the jth root, is readily obtainable.

Consider the problem of finding the later zeros of the spherical Bessel functions $J_{2m+\frac{1}{2}}(z)$ from either tabulated earlier zeros or some other suitable function of m. In the general asymptotic formula for $z_\nu^{(n)}$, the nth zero of $J_\nu(z)\cos\alpha - Y_\nu(z)\sin\alpha$, namely,

$$z_\nu^{(n)} = \left(n + \frac{1}{2}\nu - \frac{1}{4}\right)\pi - \alpha - \frac{4\nu^2 - 1}{8\{(n + \frac{1}{2}\nu - \frac{1}{4})\pi - \alpha\}} - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384\{(n + \frac{1}{2}\nu - \frac{1}{4})\pi - \alpha\}^3} - \cdots \text{[11]}, \tag{11}$$

set $\alpha = 0$ and $\nu = 2m + \frac{1}{2}$. Then from (11) it is apparent that

$$S_{n+m} \equiv (n + m)[z_{2m+\frac{1}{2}}^{(n)} - (n + m)\pi] \tag{12}$$

has a formal expansion in even powers of $1/(n + m)$, which could serve as the basis of an extrapolation formula.

However, after searching for ready-made tables of $z_{2m+\frac{1}{2}}^{(n)}$, none were found capable of testing the full potentialities of Table 4. To avoid extra labor, we shall first illustrate this principle of $(1/\nu^2)$-extrapolation with a smaller example limited to the available published 6D values of $z_{9/2}^{(n)}$ as far as $n = 6$ [12]. The problem is to calculate $z_{9/2}^{(n)}$ for $n = 6$, whose published value is 24.727566, from the four preceding values of $z_{9/2}^{(2)} = 11.704907$, $z_{9/2}^{(3)} = 15.039665$, $z_{9/2}^{(4)} = 18.301256$ and $z_{9/2}^{(5)} = 21.525418$. In other words, since $m = 2$, the problem is to find S_8 from S_4, S_5, S_6 and S_7, from which $z_{9/2}^{(6)}$ is found from (12). From (8), with $\prod_{k=4}^{10}{}'$ replaced by $\prod_{k=4}^{7}{}'$, we find $A_4(8) = -\frac{91}{3113}$, $A_5(8) = \frac{5\ 488875}{10\ 811344}$, $A_6(8) = -\frac{106883}{113264}$ and $A_7(8) = \frac{8235543}{8600448}$ from which $S_8 = \sum_{j=4}^{7} A_j(8)S_j = -3.241393$. Finally, from (12), $z_{9/2}^{(6)}$ is found to be 24.727567, which deviates by only 10^{-6} from the published value.* Comparing with $(1/\nu)$-extrapolation based upon those same values of $S_4 - S_7$, and where $A_4(8) = -\frac{1}{8}$, $A_5(8) = \frac{125}{128}$, $A_6(8) = -\frac{81}{32}$, $A_7(8) = \frac{343}{128}$, we find $S_8 = -3.241225$, from which $z_{9/2}^{(6)}$ is found to be 24.727588, which deviates by 0.000022 from the published value.

* Since we started with 6D values, it is not possible to estimate from this example the possibly higher theoretical accuracy in $(1/\nu^2)$-extrapolation, which is just the truncation error when the example is done with a sufficiently large number of places both initially and in the course of the work.

For a similar example employing Table 4, and revealing the full accuracy of (6) or (7), we choose a modification of S_{n+m}, say $\bar{S}_{n+m}$, where

$$\bar{S}_{n+m} = (n+m)[\bar{z}^{(n)}_{2m+\frac{1}{2}} - (n+m)\pi], \tag{13}$$

and where now $\bar{z}^{(n)}_{2m+\frac{1}{2}}$, instead of being the nth zero of $J_{2m+\frac{1}{2}}(x)$, is defined as a preassigned number of terms of the right member of (11) (for $\alpha = 0$, $\nu = 2m + \frac{1}{2}$) which is the same for every n. For the lowest values of n, there will be considerable deviation between the true value of the root $z^{(n)}_{2m+\frac{1}{2}}$ and the function $\bar{z}^{(n)}_{2m+\frac{1}{2}}$ which is $(n+m)\pi +$ an exact odd polynomial in $1/(n+m)$, making $\bar{S}_{n+m}$ an exact even polynomial in $1/(n+m)$. But at the inconvenience of having to compute $\bar{S}_{n+m}$ for the initial values of n, we may employ (6) or (7) to extrapolate for $\bar{S}_{n+m}$ for some larger n to get $\bar{z}^{(n)}_{2m+\frac{1}{2}}$ which will agree with the true value of the root $z^{(n)}_{2m+\frac{1}{2}}$ to very high accuracy. Taking (11) as far out as $1/\{(n + \frac{1}{2}\nu - \frac{1}{4})\pi - \alpha\}^9$, we have for $\alpha = 0$, $\nu = 2m + \frac{1}{2}$ and $\mu \equiv 4\nu^2 = (4m+1)^2$,

$$\begin{aligned}\bar{S}_{n+m} = &-\frac{\mu-1}{2^3\pi} - \frac{(\mu-1)(7\mu-31)}{3\cdot 2^7\pi^3(n+m)^2} - \frac{(\mu-1)(83\mu^2-982\mu+3779)}{15\cdot 2^{10}\pi^5(n+m)^4} \\ &- \frac{(\mu-1)(6949\mu^3-153855\mu^2+1585743\mu-6277237)}{105\cdot 2^{15}\cdot\pi^7\cdot(n+m)^6} \\ &- \frac{(\mu-1)(70197\mu^4-24\,79316\mu^3+480\,10494\mu^2 \\ &\qquad -5120\,62548\mu+20921\,63573)^*}{40320\cdot 2^{11}\cdot\pi^9(n+m)^8}.\end{aligned} \tag{14}$$

Suppose that the problem is to calculate the 14th zero of $J_{5/2}(z)$ or $z^{(14)}_{5/2}$. Then $m = 1$, and we should want to find $\bar{S}_{15}$ using Table 4 upon $\bar{S}_4 - \bar{S}_{10}$, after which we obtain $\bar{z}^{(14)}_{5/2}$ from (13).** From (14) and then (13), $\bar{z}^{(14)}_{5/2}$ which is equal to $z^{(14)}_{5/2}$ to around 14D, is found to be 47.06014 16127 6054. A quick examination of the ratios of successive terms in (14) indicates without having to compute the $1/(n+m)^{10}$ term that, to 14D, $z^{(14)}_{5/2}$ is actually 47.06014 16127 6053. Following are the calculated values of $\bar{S}_j$, for $j = n + 1 = 4(1)10$, to 16D (last figure approximate):

j	$\bar{S}_j$
4	−0.97371 85140 72535 8
5	−0.96680 12788 75286 8
6	−0.96311 73960 26803 8
7	−0.96092 04667 12625 8
8	−0.95950 42113 72512 2
9	−0.95853 75688 13022 8
10	−0.95784 82845 01448 5

Employing the older $(1/j)$-extrapolation, we find $\bar{S}_{15} = -0.95622\ 28677\ 507\ \ldots$ and from (13), $\bar{z}^{(14)}_{5/2} = 47.06014\ 16126\ 6\ \ldots$ which agrees with the true value of

* The coefficients through $1/(n+m)^6$ are from Watson [11], and the coefficient of $1/(n+m)^8$ is from Bickley and Miller [13].

** This particular problem could, of course, be set up equally efficiently computationwise by writing $\bar{S}_{n+1} = a_0 + a_1/(n+1)^2 + \cdots + a_r/(n+1)^{2r}$, where a_i is independent of n. But this present method works as long as we know somehow the values of $\bar{S}_j$.

$z_{5/2}^{(14)}$ to a unit in the 10th decimal (12th significant figure). But the $(1/j^2)$-extrapolation yields $\bar{S}_{15} = -0.95622\ 28662\ 9517\ \ldots$ and from (13), $\bar{z}_{5/2}^{(14)} = 47.06014\ 16127\ 6055\ \ldots$ which almost agrees with the true value of $z_{5/2}^{(14)}$ to 14 decimals (16 significant figures).

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Convair-Astronautics
San Diego, California

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Approximation of Curves by Line Segments

By Henry Stone

1. Introduction. Most methods of linear and nonlinear programming developed up to this time were designed to find the maximum or minimum value of a linear or nonlinear function inside a region bounded by hyperplanes. In applications of these methods it is often found that the formulation leads to constraints which are nonlinear. For example, in gasoline blending problems the relationship between lead concentration and octane number of a blend (the lead-susceptibility curve) has been found to be exponential in form. One method of dealing with such problems has been to approximate the given curves by a series of broken line segments [1]. The usual method of finding such an approximation has been, by visual examination of the graph, first to decide on the number of line segments, second, to select the intervals over which each line segment is to apply, and third, to draw in what appears to be good linear approximations to the curves over the selected intervals. Since the problem of obtaining a best fit of broken line segments to a curve does not seem to have been previously investigated, it is the purpose of this paper to formulate the problem, give a closed form solution when the given function is quadratic, show a general numerical method of solution, and apply this numerical procedure to the lead susceptibility curve.

In the case when the function is quadratic, an interesting and simple result was obtained. It was found that in fitting N lines over some interval (a, b), that the points at which one line segment was discontinued and the next line segment started were equally spaced over the interval. This result allowed the equations for the lines to be expressed in a very simple form.

2. Formulation of the Problem. Given a known nonlinear relationship

$$y = f(x), \qquad u_0 \leqq x \leqq u_N . \tag{1}$$

It is desired to obtain an approximation of the form

$$y = \begin{cases} a_1 + b_1 x, & u_0 \leqq x \leqq u_1 \\ a_2 + b_2 x, & u_1 \leqq x \leqq u_2 \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ a_N + b_N x, & u_{N-1} \leqq x \leqq u_N \end{cases} \tag{2}$$

which is best in the least squares sense. If the points $u_1, u_2, \cdots, u_N$ are specified in advance, the problem reduces to the simple case where the best fit over each interval is obtained separately by the usual least squares procedure. The results developed below will give the solution to this problem as a special case of the general solution. The more general case, treated in this paper, occurs when the end points of the line segments $u_1, u_2, \cdots, u_{N-1}$ are not specified in advance, and when only the number of line segments, N, is specified in addition to the relationship given by Equation (1).

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Following the least squares formulation, we wish to obtain the values of the variables which minimize

$$F(a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N, u_1, u_2, \cdots, u_{N-1}) = \sum_{j=1}^{N} \int_{u_{j-1}}^{u_j} (f(x) - a_j - b_j x)^2 \, dx. \tag{2}$$

The method to be used is the usual normal equation technique of finding the point at which the partial derivatives of the function (2) are zero. This procedure leads to the following system of $3N - 1$ equations to be solved for the $3N - 1$ variables $a_1, \cdots, u_{N-1}$:

$$\frac{\partial F}{\partial a_p} = 0 = -2 \int_{u_{p-1}}^{u_p} f(x) \, dx + 2a_p(u_p - u_{p-1}) + b_p(u_p^2 - u_{p-1}^2) \qquad p = 1, 2, \cdots, N \tag{3a}$$

$$\frac{\partial F}{\partial b_p} = 0 = -2 \int_{u_{p-1}}^{u_p} x f(x) \, dx + a_p(u_p^2 - u_{p-1}^2) + \tfrac{2}{3} b_p(u_p^3 - u_{p-1}^3) \qquad p = 1, 2, \cdots, N \tag{3b}$$

$$\begin{aligned} \frac{\partial F}{\partial u_p} = 0 = {} & 2f(u_p)(a_{p+1} - a_p) + 2u_p f(u_p)(b_{p+1} - b_p) \\ & - 2u_p(a_{p+1}b_{p+1} - a_p b_p) - u_p^2(b_{p+1}^2 - b_p^2) - a_{p+1}^2 + a_p^2 \end{aligned} \qquad p = 1, 2, \cdots, N - 1. \tag{3c}$$

For the moment let us assume that the values of $u_1, \cdots, u_{N-1}$ are fixed. In this case the set of Equations (3a) and (3b) are seen to be nonhomogeneous and linear in terms of the a's and b's, and, further, each a_p and b_p pair may be obtained from the solutions of N pairs of linear simultaneous equations. Carrying out this computation we obtain

$$a_p = \frac{3}{(u_p - u_{p-1})^3} \left\{ \frac{4}{3} (u_p^2 + u_p u_{p-1} + u_{p-1}^2) I(u_p, u_{p-1}) - 2(u_p + u_{p-1}) J(u_p, u_{p-1}) \right\}, \tag{4a}$$

$$b_p = \frac{6}{(u_p - u_{p-1})^3} \{ 2J(u_p, u_{p-1}) - (u_p + u_{p-1}) I(u_p, u_{p-1}) \} \tag{4b}$$

where

$$I(u_p, u_{p-1}) = \int_{u_{p-1}}^{u_p} f(x) \, dx, \qquad J(u_p, u_{p-1}) = \int_{u_{p-1}}^{u_p} x f(x) \, dx. \tag{4c}$$

Equations (4a) and (4b) represent the general solution to the problem when the points $u_1, \cdots, u_{N-1}$ have been selected in advance. When this is not the case then to proceed further we substitute the a_p and b_p from Equations (4) into Equations (3c), giving a system of $N - 1$ nonlinear equations in $u_1, \cdots, u_{N-1}$ to be solved. In general no closed form solution for this set of equations can be found. In a later

section of this report a numerical method of solution will be described which is practical for large-scale computer application. In the case when $f(x)$ is quadratic, some simplification of the problem occurs which allows a simple closed form solution to be obtained. This result is derived in the next section.

3. Solution for $f(x)$ Quadratic. In the case that $f(x)$ is quadratic, that is,

$$f(x) = rx^2 + sx + t, \tag{5}$$

the function to be minimized is

$$F = \sum_{j=1}^{N} \int_{u_{j-1}}^{u_j} (rx^2 + sx + t - a_j - b_j x)^2 \, dx. \tag{6}$$

The integrand of this function can be simplified by consolidating the linear portion of $f(x)$ with the a_j and b_j. So that, if we let

$$\begin{aligned} \alpha_j &= (a_j - t)/r \\ \beta_j &= (b_j - s)/r, \end{aligned} \tag{7}$$

Equation (6) becomes

$$F = r^2 \sum_{j=1}^{N} \int_{u_{j-1}}^{u_j} (x^2 - \alpha_j - \beta_j x)^2 \, dx, \tag{8}$$

and it is only necessary to find the minimum of Equation (8) with respect to the α_j, β_j and u_j. The constant multiplier, r^2, in (8) drops out in the derivation of the normal equations.

Following the procedure described in the last section with now $f(x) = x^2$, the α_p and β_p are obtained from Equations (4) and have the following simple form:

$$\begin{aligned} \alpha_p &= -\tfrac{1}{6}[(u_p + u_{p-1})^2 + 2u_p u_{p-1}] \\ \beta_p &= u_p + u_{p-1}, \qquad\qquad p = 1, 2, \cdots, N. \end{aligned} \tag{9}$$

Before proceeding to the solution of the general problem, let us first examine the case of $N = 2$. In this case Equations (3c) reduce to a single equation in u. Substituting the results of Equations (9) into (3c) we obtain after some straightforward but tedious algebra the following equation in u_1.

$$4(u_2 - u_0)u_1^3 - 6(u_2^2 - u_0^2)u_1^2 + 4(u_2^3 - u_0^3)u_1 - (u_2^4 - u_0^4) = 0 \tag{10}$$

which has for its only real root

$$u_1 = \tfrac{1}{2}(u_2 + u_0). \tag{11}$$

With u_1 given by (11) and α_1, β_1, α_2, β_2 by (9), it is easy to verify that

$$\alpha_1 + \beta_1 u_1 = \alpha_2 + \beta_2 u_1 \tag{12}$$

and in terms of the original variables

$$a_1 + b_1 u_1 = a_2 + b_2 u_1 ; \tag{13}$$

that is, the lines of best fit intersect at their common end point.

To proceed now with the case of fitting N lines to the quadratic, we now make

the additional continuity assumption, verified in the case $N = 2$, that each pair of adjacent lines intersect at their end points, $u_1, u_2, \cdots u_{N-1}$, so that instead of the system of nonlinear equations (3c) to solve, we have instead the system

$$\alpha_p + \beta_p u_p = \alpha_{p+1} + \beta_{p+1} u_p, \quad p = 1, 2, \cdots, N-1, \tag{14}$$

and substituting for α_p and β_p from (9) we have

$$2u_p u_{p-1} - u_{p-1}^2 = 2u_p u_{p+1} - u_{p+1}^2, \quad p = 1, 2, \cdots, N-1 \tag{15}$$

which further simplifies to

$$u_{p-1} - 2u_p + u_{p+1} = 0, \qquad p = 1, 2, \cdots, N-1. \tag{16}$$

Equation (16) may be regarded as a system of linear homogeneous difference equations, with two known boundary points, u_0 and u_N. The general solution is of the form

$$u_p = k_1 + k_2 p, \tag{17}$$

and evaluating the constants, k_1 and k_2, from the conditions at $p = 0$ and $p = N$, we have finally

$$u_p = u_0 + \frac{p}{N}(u_N - u_0), \qquad p = 1, 2, \cdots, N-1 \tag{18}$$

as the values for the common end points. This equation gives the interesting result that for the continuous case, the end points are spaced out at equal intervals over the whole interval (u_0, u_N). In applications of the method it has been found most convenient first to determine the u's from (18), next calculate the α's and β's from (9) and finally the a's and b's from

$$\begin{aligned} a_p &= r\alpha_p + t \\ b_p &= r\beta_p + s, \end{aligned} \qquad p = 1, 2, \cdots, N \tag{19}$$

where r, s, t are the coefficients of the original quadratic (5).

4. A Numerical Method of Solution. The basic problem is to find a solution of the normal equations (3); that is, to find the a's, b's and u's satisfying

$$\frac{\partial F}{\partial a_p} = 0, \qquad p = 1, 2, \cdots, N \tag{3a}$$

$$\frac{\partial F}{\partial b_p} = 0, \qquad p = 1, 2, \cdots, N \tag{3b}$$

$$\frac{\partial F}{\partial u_p} = 0, \qquad p = 1, 2, \cdots, N-1. \tag{3c}$$

It has been shown that these equations are usually so complicated that the solution cannot be obtained directly. A general method in such cases is to assume a trial solution and derive linear equations for small additive corrections. The solution of this set of linear equations leads to a new set of trial values and the process can be repeated until the corrections become negligible.

Description of the method becomes much simpler if we adopt a new notation. To this end let

$$\theta_1 = a_1, \quad \theta_2 = a_2, \quad \cdots, \quad \theta_N = a_N,$$
$$\theta_{N+1} = b_1, \quad \theta_{N+2} = b_2, \quad \cdots, \quad \theta_{2N} = b_N,$$
$$\theta_{2N+1} = u_1, \quad \theta_{2N+2} = u_2, \quad \cdots, \quad \theta_{3N-1} = u_{N-1}.$$

The system of equations to be solved is then

$$\frac{\partial F}{\partial \theta_p} = 0, \qquad p = 1, 2, \cdots, 3N-1 \tag{3'}$$

Let the trial values be denoted by $\theta_1^0, \cdots, \theta_{3N-1}^0$; expanding (3') in Taylor's series about the point θ^0, and dropping higher-order terms, we have

$$\frac{\partial F}{\partial \theta_p} = 0 \simeq \frac{\partial F}{\partial \theta_p{}^0} + \sum_{j=1}^{3N-1} (\theta_j - \theta_j^0) \frac{\partial^2 F}{\partial \theta_p{}^0 \partial \theta_j{}^0}, \qquad p = 1, \cdots, 3N-1 \tag{20}$$

where the notation $(\partial F/\partial \theta_p{}^0)$, etc. means that the indicated derivative is to be evaluated at the point θ_0. Let $\Delta\Theta = \Theta - \Theta^0$ be the vector with components, $\theta_1 - \theta_1^0, \cdots, \theta_{3N-1} - \theta_{3N-1}^0$, let G be the vector with components, $(\partial F/\partial \theta_1{}^0), \cdots, (\partial F/\partial \theta_{3N-1}^0)$, and let H be the $(3N-1)$ by $(3N-1)$ matrix whose (i, j) component is $(\partial^2 F/\partial \theta_i{}^0 \partial \theta_j{}^0)$. Then in matrix notation the right-hand side of Equation (20) becomes

$$G + \Delta\Theta \cdot H = 0 \tag{21}$$

and if H is non-singular, then Θ^1, the second approximation to the solution of (20), is given by

$$\Theta^1 = \Theta^0 - H^{-1}G. \tag{22}$$

A FORTRAN computer program for the IBM 704 has been written and tested which will solve this problem for any function, $f(x)$, which is differentiable. A FORTRAN subroutine must be supplied for each $f(x)$, which will compute $f(k)$, $I(k_1, k_2)$, $J(k_1, k_2)$ and $(\partial f(k)/\partial k)$. Further details can be supplied by the author.

5. Application to Lead Susceptibility Curves. As described in the Introduction, part of the purpose of the research described in this paper was to find the best method of linearizing the lead susceptibility curve. It has been found experimentally that for every blend examined, the relationship between octane number of the blend and quantity of tetraethyl lead added is adequately given by

$$O \cdot N \cdot = k_1 + k_2 e^{-cx} = f(x), \qquad 0 \leqq x \leqq 3.0 \tag{23}$$

where $O \cdot N \cdot$ is the octane number of the blend, x is the quantity of lead in c.c., and k_1, k_2, c are known constants representative of the blend. It has been indicated above that this problem may be treated as a single-parameter problem in c alone, since k_1 and k_2 may be absorbed into the fitted lines. Thus, instead of having the integrand of Equation (2) represented by terms of the form

$$(k_1 + k_2 e^{-cx} - a_j - b_j x)^2$$

let us set

$$\alpha_j = (a_j - k_1)/k_2$$
$$\beta_j = b_j/k_2 , \tag{24}$$

and the integrand becomes proportional to

$$(e^{-cx} - \alpha_j - \beta_j x)^2. \tag{25}$$

The method and computer program described in the last section were used to obtain values of the line and end point parameters for fitting two, three and four line segments to the exponential function, $\exp(-cx)$. The accompanying Tables, 1 through 3, give the normalized line slopes, β, the normalized intercepts, α, and the values of the end points, u, for the number of lines, N, equal to 2, 3, 4, as a function

TABLE 1
Line Parameters, $N = 2$

c	Intercepts		Slopes		Common end point, u_1
	α_1	α_2	β_1	β_2	
0.1	0.998	0.980	−0.093	−0.080	1.385
0.2	0.994	0.931	−0.174	−0.129	1.400
0.3	0.988	0.867	−0.246	−0.157	1.360
0.4	0.981	0.798	−0.309	−0.170	1.316
0.5	0.973	0.728	−0.367	−0.175	1.276
0.6	0.966	0.661	−0.420	−0.173	1.235
0.7	0.958	0.598	−0.469	−0.168	1.196
0.8	0.950	0.540	−0.515	−0.160	1.155
0.9	0.943	0.488	−0.558	−0.152	1.116
1.0	0.935	0.442	−0.600	−0.142	1.080
1.2	0.923	0.363	−0.680	−0.124	1.008
1.5	0.906	0.275	−0.793	−0.101	0.912

TABLE 2
Line Parameters, $N = 3$

c	Intercepts			Slopes			End Points	
	α_1	α_2	α_3	β_1	β_2	β_3	u_1	u_2
0.1	0.999	0.993	0.976	−0.096	−0.088	−0.079	0.785	1.783
0.2	0.998	0.975	0.919	−0.186	−0.156	−0.125	0.782	1.775
0.3	0.995	0.943	0.840	−0.266	−0.202	−0.146	0.835	1.820
0.4	0.992	0.907	0.756	−0.340	−0.237	−0.154	0.830	1.810
0.5	0.988	0.870	0.673	−0.411	−0.265	−0.153	0.801	1.773
0.6	0.985	0.835	0.595	−0.478	−0.285	−0.147	0.774	1.738
0.7	0.981	0.798	0.523	−0.541	−0.299	−0.137	0.756	1.708
0.8	0.978	0.767	0.460	−0.604	−0.312	−0.127	0.724	1.663
0.9	0.974	0.737	0.404	−0.665	−0.323	−0.117	0.694	1.618
1.0	0.971	0.711	0.355	−0.725	−0.333	−0.107	0.666	1.573
1.2	0.966	0.664	0.275	−0.841	−0.349	−0.088	0.615	1.487
1.5	0.959	0.608	0.192	−1.012	−0.371	−0.066	0.548	1.365

TABLE 3
Line Parameters, N = 4

c	Intercepts				Slopes				End Points		
	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	u_1	u_2	u_3
0.1	1.000	0.996	0.988	0.973	−0.097	−0.091	−0.085	−0.078	0.571	1.268	2.077
0.2	0.999	0.985	0.955	0.908	−0.188	−0.166	−0.143	−0.120	0.607	1.296	2.096
0.3	0.997	0.968	0.910	0.824	−0.274	−0.226	−0.181	−0.140	0.607	1.293	2.087
0.4	0.996	0.948	0.859	0.734	−0.356	−0.276	−0.206	−0.146	0.595	1.275	2.068
0.5	0.994	0.927	0.805	0.646	−0.434	−0.318	−0.221	−0.143	0.577	1.246	2.040
0.6	0.992	0.904	0.752	0.562	−0.508	−0.352	−0.228	−0.134	0.563	1.222	2.016
0.7	0.989	0.880	0.700	0.486	−0.580	−0.381	−0.231	−0.124	0.549	1.199	1.991
0.8	0.987	0.859	0.654	0.420	−0.650	−0.409	−0.232	−0.112	0.530	1.165	1.953
0.9	0.985	0.841	0.614	0.363	−0.720	−0.435	−0.233	−0.101	0.507	1.124	1.910
1.0	0.984	0.824	0.577	0.313	−0.789	−0.460	−0.232	−0.091	0.485	1.084	1.866
1.2	0.981	0.795	0.516	0.234	−0.925	−0.508	−0.230	−0.072	0.445	1.008	1.778
1.5	0.977	0.760	0.445	0.154	−1.126	−0.574	−0.227	−0.051	0.393	0.907	1.649

TABLE 4
Maximum Fitting Errors

c	$\max(e^{-cx} - \alpha_j - \beta_j x)$		
	$N = 2$	$N = 3$	$N = 4$
0.1	0.0016	0.0010	0.00055
0.2	0.0059	0.0032	0.0016
0.3	0.0119	0.0050	0.0028
0.4	0.0189	0.0080	0.0043
0.5	0.0265	0.0114	0.0062
0.6	0.0344	0.0150	0.0083
0.7	0.0423	0.0190	0.0106
0.8	0.0500	0.0223	0.0127
0.9	0.0574	0.0255	0.0145
1.0	0.0645	0.0286	0.0162
1.2	0.0774	0.0341	0.0193
1.5	0.0936	0.0410	0.0231

of the exponential parameter, c. The actual intercepts and slopes, a and b, for any particular problem are obtained from

$$a_j = k_1 + k_2\alpha_j \qquad (26)$$
$$b_j = k_2\beta_j .$$

It has been found preferable in practice, for reasons of accuracy, to determine the common end points of the lines from the relation

$$a_j + b_j u_j = a_{j+1} + b_{j+1}u_j$$

giving

$$u_j = \frac{a_j - a_{j+1}}{b_{j+1} - b_j} . \qquad (27)$$

In order to provide the user of this method with a guide to the number of line segments that should be used for any particular application, an error analysis was

carried out. The maximum difference between the exponential function, e^{-cx}, and the lines $a_j - b_j x$ were computed for the two, three, and four line cases. The results are shown in Table 4 where

$$\max(e^{-cx} - \alpha_j - \beta_j x), \qquad 0 \leqq x \leqq 3.0$$

is given as a function of c for these cases. To transform the error given in the normalized form to error in terms of octane number for any particular case, the transformation

$$\text{Error } (O\cdot N\cdot) = k_2 \cdot \text{Error (Table)}$$

is used where k_2 is the constant shown in Equation (23); that is, in the equation

$$O\cdot N\cdot = k_1 + k_2 e^{-cx}.$$

The tables given for the lead susceptibility curves can be used for curves of a more general character, namely those of the form

$$g(z) = r_1 + r_2 z + r_3 e^{-r_4 z}, \qquad w_0 \leqq z \leqq w_N. \tag{28}$$

In this case the least squares function is

$$F = \sum_{j=1}^{N} \int_{w_{j-1}}^{w_j} (r_1 + r_2 z + r_3 e^{-r_4 z} - a_j - b_j z)^2 \, dz. \tag{29}$$

Let a new variable x be defined by

$$x = 3 \cdot \frac{z - w_0}{w_N - w_0}, \qquad dx = \frac{3dz}{w_N - w_0}.$$

The range of x is $0 \leqq x \leqq 3.0$, and F becomes

$$F \propto \sum_{j=1}^{N} \int_{u_{j-1}}^{u_j} (e^{-cx} - \alpha_j - \beta_j x) \, dx \tag{30}$$

which is identical in form, except for a proportionality constant, to the previous case. The transformation equations are

$$\begin{aligned} \alpha_j &= \frac{1}{r_2}[a_j - r_1 + (b_j - r_2)w_0] \\ \beta_j &= \left(\frac{w_N - w_0}{3}\right)(b_j - r_2) \\ u_j &= 3\left(\frac{w_j - w_0}{w_N - w_0}\right). \end{aligned} \tag{31}$$

The inverse transformation equations to the original units are

$$\begin{aligned} b_j &= r_2 + \frac{3\beta_j}{w_N - w_0} \\ a_j &= r_1 + r_3\alpha_j - (b_j - r_2)w_0 \\ w_j &= w_0 + \left(\frac{w_N - w_0}{3}\right)u_j. \end{aligned} \tag{32}$$

Shell Development Company
Emeryville, California

1. W. W. Garvin, H. W. Crandall, J. B. John, R. A. Spellman, "Applications of linear programming in the oil industry," *Management Sci.* 3, 1957, p. 407.

Numerical Integration Using Sums of Exponential Functions

By F. C. Ledsham

Let y be a function of the independent variable x, and let

$$q = \frac{dy}{dx}. \tag{1}$$

Suppose that the numerical values of q be known for the $n + 1$ discrete values of x given by $x = x_0 + rh$, where r runs through the integers from zero to n and h is some fixed increment. Let $f(x)$ be some function of x chosen to coincide with q at these particular x-values. For convenience we shall denote $q(x_0 + rh)$ by q_r, with a similar convention for other functions. In order to obtain a numerical estimate of the increment of y over any chosen range of x, say $y_k - y_m$, one may then use the approximation

$$y_k - y_m = \int_{x_0+mh}^{x_0+kh} q\,dx \cong \int_{x_0+mh}^{x_0+kh} f\,dx, \tag{2}$$

the accuracy of which will depend upon the choice of the function $f(x)$. Note that k and m may have any real values (positive, negative or zero) and, in particular, are not confined to the integers nor to the region from zero to n.

It is customary to take $f(x)$ to be the lowest order polynomial satisfying the conditions specified above. As is well known, it is not then necessary to determine this polynomial, and it is possible to write

$$\int_{x_0+mh}^{x_0+kh} f\,dx = h\sum_{r=0}^{n} A(k, m, n; r)\, f_r = h\sum_{r=0}^{n} A(k, m, n; r)\, q_r, \tag{3}$$

where the A's are constants satisfying the equation

$$\sum_{r=0}^{n} A(k, m, n; r) = k - m, \tag{4}$$

and which may be tabulated once and for all as functions of k, m, n and r.

If $m = 0$ and $k = n$, then this last procedure leads to the Newton-Cotes series of formulae, of which Simpson's rule (for $n = 2$) is probably the best known. If we put $k = m + 1$, and also let this quantity equal $n + 1$ or n, then we get, respectively, extrapolation and check formulae which may be used for step-by-step numerical integration of first order differential equations. Other combinations of m and k also lead to useful formulae, and a selection of these (for various values of n) is included in a paper by Bickley [1].

It has been pointed out by Greenwood [2] and by Brock and Murray [3], that a

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polynomial form of $f(x)$ does not always provide the best approximation to $q(x)$. In particular, these authors give examples in which $f(x)$ is better taken as a sum of exponential functions, in the form

$$f(x) = \sum_{i=0}^{n} a_i \exp(\alpha_i x), \tag{5}$$

where the α_i are chosen to suit the problem concerned, and may be complex or zero.

Greenwood considers two particular forms of equation (5), both involving real values only of the α_i. For his first type he puts $\alpha_i = i$; while, for the second, he puts $\alpha_i = i - \frac{1}{2}n$ where i, in both cases, runs through the integers from zero to n. The second (symmetric) case is only applied by Greenwood to even values of n. He also takes $m = 0$ and $k = n$, to produce formulae analogous to those of the Newton-Cotes series.

Brock and Murray are concerned with the step-by-step numerical integration of first order differential equations. They consider more general cases in which the α_i are complex, and tailored to fit the particular problem to hand—with the help of earlier (and less accurate) solutions of the differential equations concerned.

The papers of Greenwood and Brock and Murray both contain practical examples of uses of the ideas expressed above, and discuss the magnitudes of the errors involved.

Using equation (5), with particular values given to the coefficients α_i, we may obtain an equation of the form

$$\int_{x_0+mh}^{x_0+kh} f\,dx = \sum_{r=0}^{n} B(k, m, n; h, r) f_r = \sum_{r=0}^{n} B(k, m, n; h, r) q_r . \tag{6}$$

Unlike the corresponding equation (3), h does not occur naturally as a factor on the right hand side of this equation, and the coefficients B have to be recalculated for every change in this quantity. Incidentally, Brock and Murray, in their work, do take out a factor h—and consequently calculate coefficients equivalent to B/h in our notation.

If one of the α_i be zero, then we have the relationship

$$\sum_{r=0}^{n} B(k, m, n; h, r) = h(k - m), \tag{7}$$

corresponding to equation (4). If none of the α_i be zero, then this last equation does not hold exactly—though it remains approximately true, and serves as a useful check during the calculation of the B's.

If one of the α_i be zero then let this be α_s. We then have

$$\begin{aligned}\int_{x_0+mh}^{x_0+kh} f\,dx &= a_s h(k - m) \\ &\quad + \sum_{\substack{i=0 \\ (i \neq s)}}^{n} \frac{a_i}{\alpha_i} [\exp\{\alpha_i(x_0 + kh)\} - \exp\{\alpha_i(x_0 + mh)\}].\end{aligned} \tag{8}$$

If none of the α_i be zero then the first term on the right hand side of this equation drops out, and the restriction $i \neq s$ is omitted from the summation.

From the $n + 1$ equations

$$f_r = q_r = \sum_{i=0}^{n} a_i \exp\{\alpha_i(x_0 + rh)\}, \tag{9}$$

the a_i, or rather $a_i \exp(\alpha_i x_0)$, may be found in terms of the q_r, enabling us to put equation (8) into the form of equation (6). In previous applications of these ideas [2, 3] this step, or its equivalent, has been carried out numerically. It is therefore considered that there may be some interest in analytical formulae into which one might substitute directly in order to obtain the required coefficients $B(k, m, n; h, r)$, such as those given below.

Write

$$\left.\begin{aligned} b_i &= a_i \exp(\alpha_i x_0), \\ t_i &= \exp(\alpha_i h) \quad , \end{aligned}\right\} \tag{10}$$

so that equations (9) may be written in the form

$$f_r = q_r = \sum_{i=0}^{n} b_i t_i^r, \qquad (r = 0, 1, \cdots n). \tag{11}$$

For any value of n, it may be verified that the solution of these $n + 1$ equations leads to the following symbolic equations for the b_i :

$$b_i = \frac{\prod_{j \neq i} (t_j - q)}{\prod_{j \neq i} (t_j - t_i)}, \tag{12}$$

where, after expansion, the powers of the q's are lowered to represent suffixes, and the term of the numerator originally independent of q is taken as the coefficient of q_0. For example, if $n = 2$ we would have

$$b_0 = \frac{(t_1 - q)(t_2 - q)}{(t_1 - t_0)(t_2 - t_0)} \quad \text{(symbolically)},$$

which would be interpreted as giving

$$b_0 = \frac{t_1 t_2 q_0 - (t_1 + t_2) q_1 + q_2}{(t_1 - t_0)(t_2 - t_0)},$$

and similarly for b_1 and b_2.

If, as in equation (8), α_s be zero then $a_s = b_s$ and that equation becomes

$$\int_{x_0 + mh}^{x_0 + kh} f \, dx = b_s h(k - m) + \sum_{\substack{i=0 \\ (i \neq s)}}^{n} \frac{b_i}{\alpha_i} (t_i^k - t_i^m). \tag{13}$$

Substituting into equation (13) the values of b_i calculated from equations (12) leads immediately to the required form (6). It will be noted that the b_i do not involve k and m, so that they do not have to be recalculated if integrations are required over ranges corresponding to more than one pair of values of these parameters.

If two of the α_i be complex conjugates, then so also will be the corresponding terms of the summation on the right hand side of equation (13). Instead of calculating both those terms completely, it is therefore only necessary to find the real part of one of them and then to double it to obtain the sum of the two.

Equations (12) and (13) have been used to recalculate some of the coefficients quoted in the papers mentioned above. The first example taken was Greenwood's

symmetrical case with $n = 6$. The second was taken from the paper by Brock and Murray, with $n = 3$ and the α_i consisting of two pairs of complex conjugates. Using the full 10-figure capacity of a desk machine, cancellation reduced the accuracy of the results obtained to some four significant figures in the first example and six in the second. The procedure given here should, however, be readily adaptable to electronic digital computers, and the increased capacity of those machines should enable the coefficients $B(k, m, n; h, r)$ to be calculated to any accuracy likely to be required in practice.

Sevenoaks
Kent, England

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2. R. E. Greenwood, "Numerical integration for linear sums of exponential functions," *Ann. Math. Statist.* v. 20, 1949, p. 608–611.
3. P. Brock & F. J. Murray, "The use of exponential sums in step by step integration," *MTAC*, v. 6, 1952, p. 63–78 and 138–150.

New Factors of Mersenne Numbers

By Edgar Karst

I have tested for prime factors all the Mersenne numbers $2^p - 1$ corresponding to prime exponents p in the interval $3000 < p < 3500$. The limit of the search for factors was $9p^2$ when no factor was previously known; otherwise the limit was $3p^2$.

The nineteen new prime factors of Mersenne numbers found by this search are displayed in the following table. Factors corresponding to smaller values of p have been listed in a paper by Brillhart and Johnson [1].

p	New factors of $2^p - 1$
3037	145 777
3041	5 565 031
3067	22 063 999
3083	15 914 447
3119	230 807·14 222 641
3121	31 509 617
3167	12 237 289
3181	127 241
3191	40 895 857
3253	46 452 841
3257	4 032 167
3299	19 873 177
3329	665 801·1 005 359·26 225 863
3391	1 519 169
3433	5 952 823·12 688 369

An extensive table by Riesel [2] includes smaller prime factors of Mersenne numbers corresponding to $p = 3037$, 3041, 3119, 3121, 3181, 3257, 3299, 3329, 3391, and 3433 in the preceding table.

Brigham Young University
Provo, Utah

1. John Brillhart & G. D. Johnson, "On the factors of certain Mersenne numbers," *Math. Comp.*, v. 14, 1960, p. 365–369.
2. H. Riesel, "Mersenne numbers," *MTAC*, v. 12, 1958, p. 207–213.

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SURVEYS AND EXPOSITORY PAPERS

A Bibliography on Approximate Integration

By A. H. Stroud

This bibliography is an attempt to provide a reference to the knowledge concerning approximate integration methods, and to make better known the recent contributions to this field. The two main subjects of this bibliography are quadrature (or numerical integration) formulas—weighted linear sums of values of the integrand (and possibly derivatives of the integrand)—and finite difference approximations. Papers dealing with error theory or other properties of these methods are included.

Some topics which are not included are: applications of integration methods to the solution of differential equations; graphical methods; and tables or rational function approximations for functions expressed as integrals.

There are several reference works which give good treatments of parts of this subject. Some of the best of these—HILDEBRAND [1], BUCKINGHAM [1], and MINEUR [3]—give good summaries of the well-known integration methods for functions of one variable.

The recent book by V. I. KRYLOV [7] is probably the best all around book to date concerned with approximate integration for functions of one variable.

The small book by NIKOL'SKII [3] treats "best" quadrature formulas and extremal problems related to quadrature formulas.

The two papers by HAMMER [2] and STROUD [3] together give a good survey of what is known concerning quadrature formulas for functions of more than one variable.

KANTOROVICH & KRYLOV [1] give a good discussion concerning methods for the approximate solution of integral equations.

For references to works giving quadrature formulas, interpolation tables or tables of integrals, see FLETCHER, MILLER & ROSENHEAD [1].

A few references of historical importance have been included here. A more complete list of papers prior to 1864 is given by BIERENS DE HAAN [1], who gives 60 papers concerned with approximate evaluation of integrals.

The titles of the foreign language papers, other than French, Spanish and German, usually have been translated into English. Included are the volume, page references to reviews in *Mathematical Reviews* (MR) and *Zentralblatt für Mathematik* (Zbl.).

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The journal abbreviations are those given in *Mathematical Reviews*, Index for Volume 19, pages 1417–1430.

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University of Wisconsin
Madison, Wisconsin

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

1[A–F, G–I, K, M]—NINA M. BURUNOVA, *Spravočnik po matematičeskim tablitsam, Dopolnenie N.* 1 (*Handbook on Mathematical Tables, Supplement No. 1*), Izdatel'stvo Akad. Nauk SSSR, Moscow, 1959, 184 p., 26 cm. Price 9 rubles.

This is a first supplement to the important work by Lebedev and Fedorova [1], which appeared in 1956. The arrangement is similar to that of the original, with a Part I in fifteen chapters listing the contents of the various tables, which are referred to by number, and a Part II giving the numbered references, separately for each chapter.

With two small exceptions, the titles of the chapters are unchanged. The translated heading of Chapter IV, formerly "Decimal and natural logarithms," is now simply "Logarithms"; this allows the inclusion of a few tables of logarithms to base 2. Chapter XV, besides the former "Prime numbers, factors, products, quotients and fractions," now covers also "Conversion from one system of numeration to another"; this accommodates a few tables which deal with binary-decimal and similar conversions. A three-page list of functions, giving references to both *Handbook* and *Supplement*, also shows a few additional categories not involving alteration of chapter headings.

As in the original Part II, references to works in Russian are naturally given in Cyrillic characters, but the far more numerous references to other works are given in roman characters. A mathematician knowing no Russian could probably contrive to use the greater part of the book.

Like the original work, the supplement is very welcome.

A. F.

1. A. V. LEBEDEV & R. M. FEDOROVA, *Spravočnik po matematičeskim tablitsam*, Moscow, 1956. See RMT 49, *MTAC*, v. 11, 1957, p. 104–106.

2[A–E, G, J, M, Q]—FRIEDRICH O. RINGLEB, *Mathematische Formelsammlung*, (7th ed.) Sammlung Göschen Band 51/51a, Walter de Gruyter & Co., Berlin, 1960, 320 p., 16 cm. Price DM 3.60.

This little German volume, from the *Sammlung Göschen*, is an enlarged version (42 extra pages, 3 extra figures) of the 1956 edition written by O. Th. Bürkeln. It is a classified collection of formulae and standard theorems from all branches of (essentially) undergraduate mathematics. As such, there is some, but not too much, overlap with similar collections, such as the well-known *Smithsonian Mathematical Formulae*. It is entirely different, however, from the numerical analysis handbook in the Sammlung Göschen, *Formelsammlung zur praktischen Mathematik*, by Günther Schulz.

One (probably not anticipated) use to students of this little book is as a pleasant means of picking up a basic mathematical German vocabulary. The sentences are short, factual, and not unduly encumbered by grammar or philosophy.

The sixteen chapter headings are: Arithmetik und Kombinatorik, Algebra, Zahlentheorie, Elementare Reihen, Ebene Geometrie, Stereometrie, Ebene Trigonometrie, Sphärische Trigonometrie, Mathematische Geographie und Astronomie,

Analytische Geometrie der Ebene, Analytische Geometrie des Raumes und Vektorrechnung, Differentialrechnung, Integralrechnung, Funktionentheorie und konforme Abbildung, Differentialgeometrie, and Differentialgleichungen.

D. S.

3[C, L]—L. K. Frevel & J. W. Turley, "Seven-Place Table of Iterated $\log_e (1 + x)$," The Dow Chemical Company, Midland, Michigan, 1960. Deposited in UMT File.

The n-fold iteration of $\log_e (1 + x)$ is explicitly given by

$$x - \frac{nx^2}{2} + \sum_{t=3}^{\infty} \left\{ (-1)^{t-1} n \left[\frac{1}{t} + \frac{n-1}{2} \sum_{a=0}^{t-3} C_a n^a \right] x^t \right\},$$

where the C_a's are numerical rational fractions. Using nine terms of this expansion the authors have compiled a 7D table containing 4000 different entries for $n = 0(.05)10$ and $x = 0(.05)1$. All computations were programmed on the DATATRON 220, and the output in tabular format was printed directly by a Teletype printer. The recorded values are accurate to $\pm 4 \cdot 10^{-7}$.

Authors' Summary

4[F]—C. L. Baker & F. J. Gruenberger. *The First Six Million Prime Numbers*, The RAND Corporation, Santa Monica, published by The Microcard Foundation, Madison, Wisconsin, 1959, 8 p., 16 x 23 cm. + 62 cards, 7.5 x 12.6 cm. Price $35.00.

This unusual table is arranged on microcards which present 124 photographs. Each photograph (except the first and last) displays 39 pages of tabulation. Each page lists 1250 prime numbers. The primes range from 1, which is counted as a prime, to 104395289. Each line of a page contains 25 consecutive primes. The first prime in the line is given completely; only the last three digits of the other 24 primes are given. The rank of a prime, once it is located in the table, is given by an obvious formula in terms of its page number, line number, and position in that line. It is just as easy to find isolated values of $\pi(x)$, the number of primes $\leqq x$.

It is obvious that the very high condensation of information achieved in this list of primes is not won without some difficulty, namely, the fact that it is invisible to the naked eye. A quite strong pocket magnifying glass or a microcard reading machine is required to read the data. Any use of the table other than finding whether or not a given number is prime or evaluating $\pi(x)$ for isolated values of x is really impractical. For example, to determine the number of twin primes in the 56th million or to calculate a sum involving consecutive primes—problems in which one must carefully keep one's place in the table—would be difficult indeed. Of course, such problems should be done by an electronic digital computer anyway.

There is an interesting description of the IBM 704 program used to generate the list of primes on punched cards, which also contain the differences between consecutive primes.

D. H. Lehmer

University of California
Berkeley, California

Editorial Note.—If 2 is counted as the first prime, which is the current practice, then the six millionth prime is 104395301, which happens to be the first member of a prime pair.

5[F].—H. GUPTA, M. S. CHEEMA, A. MEHTA & O. P. GUPTA. Edited by J. C. P. MILLER, *Representation of Primes by Quadratic Forms*, Royal Society Mathematical Tables No. 5, Cambridge University Press, New York, 1960, xxiv + 135 p., 29 cm. Price $8.50.

The eight tables in this work give solutions a, b and k, n to two related Diophantine equations

$$\kappa p = a^2 + Db^2 \quad \text{and} \quad \kappa kp = n^2 + D,$$

with $\kappa = 1$ and 2, $D = 5$, 6, 10, and 13, and for all primes $p < 100{,}000$ for which solutions exist. In each table there are approximately 2400 primes, that is, about one-fourth of all primes $<100{,}000$.

The subtitle indicates that this volume is Part I of a larger work and implies that other values of D will be forthcoming. The significance of these particular values, 5, 6, 10, and 13, is that in all quadratic number fields, $R(\sqrt{-D})$, these are the smallest positive D's for which unique factorization of the algebraic integers is lacking. In all four cases the class number is 2 (there are two classes of ideals) and this is associated with the two values of κ. For about one-half of the p's for which $-D$ is a quadratic residue a solution exists for $\kappa = 1$, and for the remaining one-half, for $\kappa = 2$. It may be noted that the $\kappa = 1$ table is always somewhat shorter than its companion $\kappa = 2$ table. This is as expected, since there are *generally* more primes of the form $4Dm + N$ than of the form $4Dm + R$ if R is a quadratic residue of $4D$ and N is not (see *MTAC*, v. 13, 1959, p. 272–284).

There is an interesting, twelve-page introduction to the background ideal and class number theory. It is boldly stated there, p. xii, that unique factorization exists for square-free $D > 0$ only if $D = 1$, 2, 3, 7, 11, 19, 43, 67, and 163. However, this has never been fully proven. This background theory culminates in several theorems due to H. P. F. Swinnerton-Dyer.

The tables were done by hand, "with the help of two long strips of paper—A and B," and were carefully checked in a variety of ways. Why the solutions to $kp = n^2 + 6$ for $p = 7$ and $p = 31$ are listed, on p. 36, as k, $n = 6$, 6 and 22, 26 respectively rather than the obvious 1, 1 and 1, 5 is not clear to the reviewer. But this slip seems to be exceptional.

The tables may be used as lists of prime ideals in the four fields, $R(\sqrt{-D})$. The dust jacket suggests a second application, that they "may contribute to the understanding of such unsolved questions as 'Is the number of primes of form $n^2 + 5$, or $n^2 + 6$, etc., infinite or finite?' " In this respect, however, it may be remarked that the data here are rather meager since the primes are $<10^5$, while in 10 minutes an IBM 704 can count such primes up to $3.24 \cdot 10^{10}$ (see *MTAC*, v. 13, 1959, p. 78–86). In fact the 16th prime of the form $n^2 + 5$ is 86441 and is the largest listed in this volume, while the 4368th prime of that form is 32,371, 926, 089. Nonetheless, the data are sufficient to indicate at least rough agreement with the Hardy-Littlewood conjecture. Since

$$L_5(1) = 1.405 > L_6(1) = 1.283 > L_{10}(1) = 0.993 > L_{13}(1) = 0.871$$

we should expect the relative density of primes to increase as we progress from $n^2 + 5$ to $n^2 + 6$ to $n^2 + 10$ to $n^2 + 13$ (*Math. Comp.*, v. 14, 1960, p. 324–326). This is indeed the case.

An apparent anomaly concerned $n^2 + 13$, where there were numerous primes from $n = 0$ to 68 and from $n = 264$ to 298, but none in between. This striking maldistribution was most alarming, and threatened dire consequences to the Hardy-Littlewood conjecture, until the real explanation was found—pages 105 to 120 were missing in the reviewer's copy. Aside from this gross lapse, the volume has the usual elegance of the Royal Society Mathematical Tables.

D. S.

6[F, L].—C. B. HASELGROVE in collaboration with J. C. P. MILLER, *Tables of the Riemann Zeta Function*, Royal Society Mathematical Tables No. 6, Cambridge University Press, New York, 1960, xxiii + 80 p., 29 cm. Price $9.50.

These important and fascinating tables are concerned primarily with $\zeta(\frac{1}{2} + it)$, the zeta function for real part $\frac{1}{2}$, and with its zeros. This complex function is expressed both in cartesian and polar forms:

$$\begin{aligned}\zeta\left(\tfrac{1}{2} + it\right) &= \mathfrak{R}\zeta\left(\tfrac{1}{2} + it\right) + i\mathfrak{I}\zeta\left(\tfrac{1}{2} + it\right) \\ &= Z(t)e^{-i\theta(t)}.\end{aligned}$$

In the latter, $\theta(t)$ is continuous, with $\theta(0) = 0$, and the signed modulus $Z(t)$, given by

$$Z(t) = \pi^{-\frac{1}{4}it}\frac{\Gamma(\frac{1}{4} + \frac{1}{2}it)}{|\Gamma(\frac{1}{4} + \frac{1}{2}it)|}\zeta\left(\tfrac{1}{2} + it\right),$$

changes sign at every zero. The nth zero, γ_n, is the nth solution of $Z(\gamma) = 0$ and the nth Gram point, g_n, is the solution of $\theta(g_n) = n$.

Table I gives $\mathfrak{R}\zeta(\frac{1}{2} + it)$, $\mathfrak{I}\zeta(\frac{1}{2} + it)$, $Z(t)$, and $\theta(t)$ to 6D for $t = 0(0.1)100$. $\mathfrak{R}\zeta(1 + it)$ and $\mathfrak{I}\zeta(1 + it)$ are also listed.

Table II gives $Z(t)$ to 6D for $t = 100(0.1)1000$.

Table III has two parts. Part 1, for $n = 1(1)650$, gives γ_n, g_{n-1}, and $\phi_n = (1/\pi)ph\zeta'(\frac{1}{2} + i\gamma_n)$ to 6D and $|\zeta'(\frac{1}{2} + i\gamma_n)|$ to 5D. Part 2, for $n = 651(1)1600$, gives γ_n to 6D and $|\zeta'(\frac{1}{2} + i\gamma_n)|$ to 5D.

Table IV gives $Z(t)$ to 6D for four other ranges of t,

$$t = 7000(0.1)7025, \quad 17120(0.1)17145, \quad 100000(0.1)100025, \quad 250000(0.1)250025.$$

Also for these four ranges are given the zeros to 6D and derivatives to 5D. There are 28, 32, 38, and 42 zeros in the four ranges respectively.

Table V gives $(1/\pi)ph\Gamma(\frac{1}{2} + it)$ to 6D for $t = 0(0.1)50(1)600(2)1000$.

The inclusion of g_n and ϕ_n in part 1 of Table III allows the reader to study Gram's "Law" which states that the zeros and Gram points are interlaced:

$$\gamma_{n-1} < g_{n-2} < \gamma_n < g_{n-1} < \gamma_{n+1}.$$

A violation occurs if $|\phi_n| > \frac{1}{2}$. The first violation is for $n = 127$. The first double violation is for $n = 379$ and 380—i.e., there are three Gram points between γ_{379} and γ_{380}. In all, there are 22 violations in these 650 zeros. Gram's "Law" may also be expressed by saying that the complex $\zeta(\frac{1}{2} + it)$ approaches its zero via the 4th or 3rd quadrant. Thus the following statistics for these 650 zeros are of interest: 4th quadrant, 320 cases; 3rd quadrant, 308; 2nd quadrant, 13, and 1st quadrant, 9.

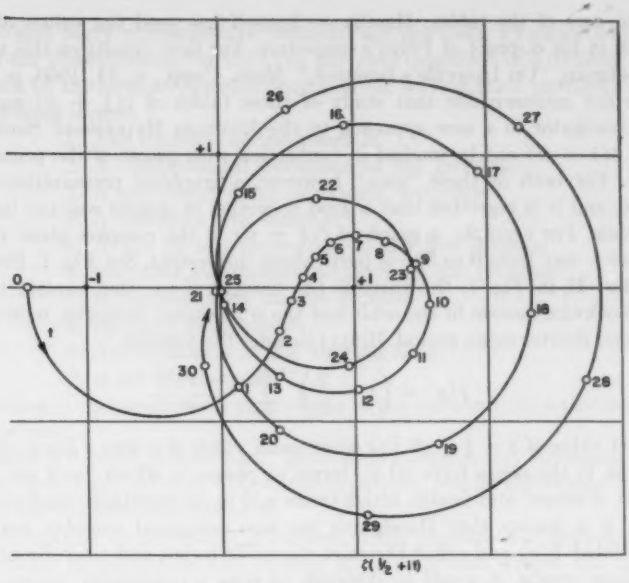

FIG. 1.—$\zeta\ (\frac{1}{2} + it)$ in the complex plane, for $0 \leqq t \leqq 30$.

This suggests the conjecture* that $\operatorname{Lim}_{N\to\infty} (1/N) \sum_1^N \phi_n = 0$, that is, that

$$\zeta'(\tfrac{1}{2} + i\gamma_n)$$

is positive real in the mean. Since violations against the "Law" are associated with exceptionally close zeros or exceptionally large values of $Z(t)$, some extremes discovered by D. H. Lehmer are of interest, and are listed in Table IV:

$$\gamma_m = 7005.062\ 866 \quad \text{and} \quad \gamma_{m+1} = 7005.\ 100\ 565 \qquad (m = ?)$$

$$Z(17123.1) = 18.955\ 257.$$

The tables of $\zeta(\frac{1}{2} + it)$ and of $Z(t)$ were both computed on two different machines, EDSAC and the Manchester Mark I, from two different asymptotic formulae, one due to Gram, the second a modification of the Riemann-Siegel formula due to Lehmer. A great deal of difficulty was experienced in eliminating discrepancies between the two methods, and three or four erroneous tables had to be discarded.

* *Note added November 10, 1960.* For the first 650 zeros this mean value, $\bar{\phi}_n$, shows such a *marked* trend toward zero that it is even probable that $|\Sigma\,\phi_n| / \sqrt{N}$ also tends toward zero. Taking only the "worst" points—that is, where $|\Sigma\,\phi_n|$ reaches a new maximum—sample values of $|\Sigma\,\phi_n| / \sqrt{N}$ are .0513, .0437, .0380, .0328, .0279, .0251, and .0217 for n = 8, 33, 64, 126, 256, 379, and 606, respectively. It should be noted that $\bar{\phi}_n$ oscillates as it tends towards zero —up to n = 650 it changes sign 265 times. In particular, all 22 of the Gram violations, $|\phi_n| > \frac{1}{4}$, are associated with such sign changes. But this latter is almost inevitable up to n = 650, since the extreme values of $\Sigma\,\phi_n$ up to this limit are only +.520578 for n = 567 and −.533885 for n = 606.

As to uses of the tables, Haselgrove himself has used the values of the first 600 zeros in his disproof of Pólya's conjecture. For later results on this theme, see R. S. Lehman, "On Liouville's function," *Math. Comp.*, v. 14, 1960, p. 311–320.

It is not inconceivable that study of these tables of $\zeta(\frac{1}{2} + it)$ may inspire some investigator to a new approach to the Riemann Hypothesis. Similarly, the table of $\zeta(1 + it)$ can be studied in connection with proofs of the prime number theorem. For both of these "uses," however, a graphical presentation is highly desirable, and it is regretted that a good collection of graphs was not included in this volume. For example, a graph of $\zeta(\frac{1}{2} + it)$ in the complex plane versus the parameter t—say from 0 to 30—is particularly interesting. See Fig. 1. Problem for the reader: If, in Fig. 1, the variable t is thought of as time, explain the initial counterclockwise motion in the orbit and the subsequent clockwise motion with a shorter and shorter mean period. Hint: Consider the formula

$$\zeta(s) = \left(1 - \frac{2}{2^s}\right)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

for a fixed value of $s = \frac{1}{2} + it$. For what mean value of n does a block of consecutive terms in the series have all its terms in phase; in which block are alternate terms out of phase; and finally, which terms add in an essentially random manner?

Since it is known that Haselgrove has also computed complex zeros of the closely related $L(s)$ and other Dirichlet character series, and since he has not included these results, it would be desirable to issue a companion volume to make these related tables generally available.

Two very minor errors were noted in the Introduction.

Page xii, line 5: change $\zeta(\rho_n)$ to $\zeta'(\rho_n)$.

Page xvi, line 6 from bottom: change (3.7) to (3.32).

With respect to Liénard's table of $\zeta(n)$ for $n = 2(1)167$, which is mentioned on page xx, a recent extension should be noted: J. W. Wrench, Jr., "Further evaluation of Khintchine's constant," *Math. Comp.*, v. 14, 1960, p. 370–371.

D. S.

7[G, K, X, Z].—Anthony Ralston & Herbert S. Wilf, *Mathematical Methods for Digital Computers*, John Wiley & Sons, Inc., New York, 1960, xi + 293 p., 27 cm. Price $9.00.

This book contains contributions from twenty-four research workers in numerical analysis and related fields. Two of the contributors serve as editors, and, as the title implies, all contributors are definitely high-speed-computer oriented. It is interesting to note that ten of the twenty-four authors are from universities.

In the introduction it is stated that "the major purpose of this book is to present many—but by no means all—of the more commonly used tools of the modern numerical analyst along with some of the more promising newly developed methods. The motivation behind this presentation is not only to gather together in one place a partial survey of modern numerical methods but also, in each case, to acquaint the reader with the interplay between computer capabilities and processes of analysis".

The editors have done a good job in carrying out the purpose of the book. They have coordinated the efforts of their colleagues in a remarkable way. About twenty

important and fairly representative problems are discussed by experts in various fields and the presentations, for the most part, are organized in a uniform manner. All but one of the authors describe programs for solving their problems in terms of the following topics:

1. The function of the program
2. A mathematical discussion of the problem
3. A summary of the calculation procedure
4. A complete flow diagram
5. A box by box description of the flow diagram
6. Standard subroutines required by the program
7. A representative sample problem
8. Memory requirements
9. An estimation of the running time
10. A list of references

Actual coding of the programs is omitted, as is any reference to a particular computing machine.

The book is divided into six parts: generation of elementary functions, matrices and linear equations, ordinary differential equations, partial differential equations, statistics, and miscellaneous methods.

Part I contains an interesting discussion of the generation of elementary functions by means of polynomial and rational approximations. Ten of the more common functions are examined in detail, and comparisons of the errors obtained using various approximation methods are included. Because of the nature of its subject matter this part has a format different from that discussed in Paragraph 3. Part II contains six chapters covering matrix inversion, systems of linear equations, and the matrix eigenvalue problem for symmetric matrices. Part III contains four chapters concerned with the numerical solution of ordinary differential equations, and Part IV contains five chapters discussing parabolic, elliptic and hyperbolic partial differential equations. There are four chapters in Part V covering multiple regression analysis, factor analysis, autocorrelation and spectral analysis, and the analysis of variance. The six chapters in Part VI discuss methods for numerical quadrature, Fourier analysis, linear programming, network analysis, and the solution of polynomial equations.

This book should serve as an excellent reference work for those workers who need to solve problems using electronic digital computers. For this reason the format, described above, is especially good. The book is not intended to be a textbook for introducing the subject of numerical analysis, but most students of the subject will find useful information here. Lists at the end of the various chapters contain over two hundred references relating to the problems under consideration.

The reviewer would like to see a supplementary volume appear containing many of the important methods not covered in the present volume. For instance, Part II does not cover the Givens method for finding the eigenvalues of symmetric matrices. Omitted entirely is any discussion of methods for finding the eigenvalues of arbitrary matrices. Other readers will find additional important methods they would like to see covered in a second volume.

ROBERT T. GREGORY

The University of Texas
Austin, Texas

8[G, Z].—FRANZ E. HOHN, *Applied Boolean Algebra, An Elementary Introduction*, The Macmillan Company, New York, 1960, xx + 139 p. $2.50 (paperback, offset print).

The author gives as much information as it seems possible to include conveniently in a text of this length starting with the veriest elements concerning the use of Boolean algebra in the design of switching circuits—especially for digital computers.

Examples of engineering applications are given, but no complete design of a very extensive arithmetic unit is undertaken. Hence, the ultimate dependence of the designers of computers on Boolean algebra is not completely illustrated, although it is strongly and correctly implied.

The crux of any applications of Boolean algebra to the design of switching circuits lies in computational schemes for writing fairly efficient statements of Boolean propositions. This problem is faced by the author, but only to an extent which permits the reader (and problem worker) to understand the nature of the difficulties which are encountered and some of the procedures which promise to be helpful. In this regard the pamphlet is no less informative than most of the other textbook material available, but additional reference to computational efforts would have been welcome.

The electrical elements to be used are described abstractly in a reasonable way, and altogether the presentation is self-contained, lucid, and reasonably illustrated by problems. No sophistication is required in the reader except for motivation.

Attention is not restricted to circuit design, and the standard applications of Boolean algebra are treated to an extent which is indicated in the chapter headings listed below:

Introduction
Boolean Algebra as a Model of Combinational Relay Circuitry
Boolean Algebra as a Model of Propositional Logic
The Boolean Algebra of the Subsets of a Set
The Minimization Problem
The Binary System of Numeration (Appendix I)
Semiconductor Logic Elements (Appendix II)

It would have been helpful if the author had included various alternate notations. He uses $\wedge$ for "or" and no symbol for "and." A short table of notations would be helpful to the neophyte, for not all authors have the thoughtfulness to describe their notation. The bibliography is not extensive.

The printing is by photographic offset process from typed copy, and there was a considerable amount of smearing and a number of ghost images in the review copy. However, these defects did not make reading seriously difficult.

This pamphlet should be a handy introduction to Boolean algebra for many users and a useful adjunct to the texts for several courses which might be offered in colleges.

C. B. TOMPKINS

University of California
Los Angeles 24, California

9 [K].—B. M. BENNETT & P. HSU, *Significance Tests in a 2 × 2 Contingency Table: Extension of Finney-Latscha Tables*, July 1960. Deposited in UMT File.

In testing the significance of deviations from proportionality in a 2 × 2 con-

tingency table

	With Attribute	Without Attribute	Total
Series I	a	$A - a$	A
Series II	b	$B - b$	B
Total	r	$N - r$	N

with fixed marginal totals $A \geqq B$, and $a/A \geqq b/B$, Finney (1948, *Biometrika* 35, 145) supplied the exact probabilities and the significant values of b at levels .05, .025, .01, .005, respectively for $A = 3(1)\ 15$. An extension of the same table for $A = 16(1)\ 20$ was prepared by Latscha (1953, *Biometrika*, 40, 74). The present tables further extend the exact probabilities (4 decimal places) to the case: $A = 21(1)\ 25$ and the significant b values at the four levels of the Finney-Latscha tables.

AUTHORS' SUMMARY

10[K].—MORDECAI EZEKIEL & KARL A. FOX, *Methods of Correlation and Regression Analysis*, 3rd edition, John Wiley & Sons, Inc., New York, 1959, xv + 548 p., 24 cm. Price $10.95.

It is very pleasing to see that Dr. Ezekiel's well-known text, *Methods of Correlation Analysis*, has been modernized. The older excellent book, which for many years stood alone in the desert of statistical literature, has now been joined by a growing array of fine statistical text material. Professor Fox was clearly a wise choice for Dr. Ezekiel to make for the co-author of the present edition.

This book is aimed mainly at the non-mathematical reader, in that algebraic and computational methods are stressed. There is also a wealth of practical material drawn from applied research. However, the mathematical statistician will also find much of great value; for example, the use of digital computers for data analysis is something which not all statisticians are yet well aware of.

In general, the treatment of regression and correlation is quite comprehensive, and results from the recent theoretical literature have been utilized throughout the book. As an example of this point, mention may be made of the inclusion of material dealing with regression and the analysis of variance, time series and errors, and the fitting of simultaneous relations.

The book is divided into seven main sections, with twenty-six chapters in all. The main sections are:

1. Introductory Concepts
2. Simple Regression, Linear and Curvilinear
3. Multiple Linear Regressions
4. Multiple Curvilinear Regressions
5. Significance of Correlation and Regression Results
6. Miscellaneous Special Regression Methods
7. Uses and Philosophy of Correlation and Regression Analysis

There are also three appendices, providing a glossary and important equations, methods of computation, and some technical notes. The inclusion of an author index is also a commendable feature.

Those working in the field of economics, agriculture, and business statistics will find this text of much value, since a good deal of the material is slanted toward

those areas. For teachers, statisticians in general, and members of operations research teams, this revised edition will be found to fill a real need. For applied work in fields like engineering, medicine, sociology, and psychology, the full treatment of the basic concepts of regression and correlation will be of immense value, and some slight reconsideration of the illustrative examples given will often provide insight into the real problems in these other fields.

Thanks are due to both the authors and publishers for making this material available. It is a text that should be in the library of every technical organization.

HARRY WEINGARTEN

Special Projects Office
Department of the Navy
Washington, D. C.

11[K].—E. FIX, J. L. HODGES & E. L. LEHMANN, "The restricted chi-square test," included in *Probability and Statistics*, edited by U. GRENANDER, Almqvist & Wiksell, Stockholm; John Wiley and Sons, New York, 1959, pages 92–108 [See the following review].

This paper contains some new tables of the power function of chi-square tests, i.e., of the non-central chi-square distribution, for small degrees of freedom. As is well-known, the power function $\beta = \beta(\alpha, f, \lambda)$ of a chi-square test depends on three parameters: α, the "level of significance" at which the test of the null hypothesis H_0 is conducted, i.e., the probability of the test falsely rejecting H_0 when it is true; f, the "degrees of freedom" of the test; and λ, the "non-centrality parameter," which measures the "distance" of the alternative $H = H(\lambda)$ under consideration, from the null hypothesis H_0. The tables of this article give λ to 3D as a function of $\beta = 0.5(0.1)0.9, 0.95$, for $f = 1(1)6$ and $\alpha = 0.001, 0.005, 0.01, 0.05(0.05)0.3, 0.4, 0.5$. The quantity tabulated is that value of the parameter λ which satisfied the equation

$$e^{-(\lambda/2)} \sum_{k=0}^{\infty} \frac{1}{k!\, 2^{(1/2)f+2k-1}\Gamma(f/2)} \int_{\chi_f(\alpha)}^{\infty} x^{f+2k-1} e^{-(1/2)x^2}\, dx = \beta$$

where f = number of degrees of freedom and $\chi_f(\alpha)$ is such that

$$\frac{1}{2^{(1/2)f-1}\Gamma(f/2)} \int_{\chi_f(\alpha)}^{\infty} x^{f-1} e^{-(1/2)x^2}\, dx = \alpha.$$

These tables thus supplement those of E. Fix (1949), reviewed in *MTAC*, v. 4, 1950, p. 206–207.

CHURCHILL EISENHART

National Bureau of Standards
Washington, D. C.

12[K].—ULF GRENANDER, Editor, *Probability and Statistics*, Almqvist & Wiksell, Stockholm; John Wiley & Sons, New York, 1959, 434 p., 24 cm. Price, $12.50.

"Once it had been suggested that a book of studies in probability and statistics should be presented to Harald Cramér in honor of his 65th birthday, the authors

needed little or no persuasion to contribute." This volume, subtitled The Harald Cramér Volume, is the result. It constitutes a fitting tribute to Harald Cramér, Professor of Mathematical Statistics and Actuarial Mathematics at the University of Stockholm for over three decades, then President, and now Chancellor of the University of Stockholm—Sweden's outstanding figure in the mathematical theory of probability, mathematical statistics, and actuarial mathematics.

The twenty-one contributors to this volume are from six countries—England (2), Finland (1), France (1), India (1), Sweden (3), United States (13); five are former students of Professor Cramér; and all have enviable international reputations in their respective fields. Their respective contributions are arranged within the volume in alphabetical sequence, with page-lengths indicated in parentheses, and are as follows: T. W. Anderson, "Some scaling models and estimation procedures in the latent class model" (30); M. S. Bartlett, "The impact of stochastic process theory on statistics" (11); J. L. Doob, "A Markov chain theorem" (8); G. Elfving, "Design of linear experiments" (17); W. Feller, "On combinatorial methods in fluctuation theory" (17); E. Fix, J. L. Hodges and E. L. Lehmann, "The restricted chi-square test" (16); U. Grenander, "Some non-linear problems in probability theory" (22); M. Kac, "Some remarks on stable processes with independent increments" (9); D. G. Kendall, "Unitary dilations of Markov transition operators, and the corresponding integral representations for transition-probability matrices" (23); P. Levy, "Construction du processus de W. Feller et H. P. McKean en partant du mouvement Brownien" (13); P. Masani, "Cramér's theorem on monotone matrix-valued functions and the Wold decomposition" (15); P. Masani and N. Wiener, "Non-linear prediction" (23); J. Neyman, "Optimal asymptotic tests of composite statistical hypotheses" (22); H. Robbins, "Sequential estimation of the mean of a normal population" (11); M. Rosenblatt, "Statistical analysis of stochastic processes with stationary residuals" (30); C. O. Segerdahl, "A survey of results in the collective theory of risk" (24); J. S. Tukey, "An introduction to the measurement of spectra" (31); S. S. Wilks, "Non-parametric statistical inference" (24); H. Wold, "Ends and means in econometric model building" (80). Only one of these papers (E. Fix et al) contains a mathematical table of general interest, which is considered separately in the immediately preceding review.

All in all, this volume provides a panoramic and stimulating view of the work at the frontiers of probability and statistical theory and some of their applications. The individual scientist, unless he is intimately concerned with the theory of stochastic processes, will very likely find only a small fraction of its contents of direct interest to him, and, therefore, may consider the volume much too expensive for personal acquisition. On the other hand, it *is* the type of volume that one expects to find in the library of a university where research in probability and mathematical statistics and their applications is conducted at the post-graduate level, and in the libraries of other organizations where research is carried out in the above and related fields.

Finally, it is to be regretted that the volume does not contain either a photograph or a biography of Professor Cramér at this milestone in his career.

CHURCHILL EISENHART

National Bureau of Standards
Washington, D. C.

13[K].—MASAAKI SIBUYA, "Modal intervals for chi-square distributions," *Ann. Inst. Statist. Math.*, v. 9, 1958, p. 225–236.

Let $f(x)$, $-\infty < x < \infty$, be a continuous unimodal probability density function, and let L, $U (L < U)$ satisfy the conditions: $\int_L^U f(x)\,dx = 1 - \alpha$ and $f(U) = f(L)$. Then $|L, U|$ is called the $(1 - \alpha)$-content modal interval for $f(x)$. In this paper $f(x)$ is taken to be the χ^2 density function with ϕ degrees of freedom. Applications are discussed, and a method of computation of modal intervals with given content is developed for this case. Tables are presented of L to 4D, U to 3D, $P_r(\chi^2 \leqq L)$ to 5D, and $(U - L)/(U + L)$ to 4D, for $\phi = 3(1)30(10)100$ and $1 - \alpha = .8, .9, .95, .99$. A previous table [1] gave values of $1 - \alpha$, effectively for $(U - L)/U + L) = .01, .05, .10, .20$, and $\phi = 4(4)80$.

C. C. CRAIG

University of Michigan
Ann Arbor, Michigan

1. M. A. GIRSCHICK, H. RUBIN & R. SITGREAVES, "Estimates of bounded relative error in particle counting," *Ann. Math. Statist.*, v. 26, 1955, p. 276–285.

14[K].—MASAAKI SIBUYA & HIDEO TODA, "Tables of the probability density function of range in normal samples," *Ann. Inst. Statist. Math.*, v. 8, 1957, p. 155–165.

This paper gives details of the calculation of the probability density function $f_n(w)$, for which there are included tables to 4D corresponding to $w = 0(.05)7.65$ and $n = 3(1)20$, where n represents the size of the normal sample. Cadwell's formula [1] is cited as the basis for this calculation.

W. J. DIXON

University of California
Los Angeles, California

1. J. H. CADWELL, "The distribution of quasi-ranges in samples from a normal population," *Ann. Math. Statist.*, v. 24, 1953, p. 603–613.

15[K, S, V].—O. G. SUTTON, *Mathematics in Action*, Harper & Bros., New York, 1960, xvi + 236 p., 20 cm. Price $1.45 (Paperback).

In the words of its author, "This book, written primarily for the layman, will prove . . . of interest also to students, especially those in the upper forms of schools or in the first years in the university. It is a view of the part played by mathematics in applied science, as seen by a mathematical physicist."

Chapter 1, "The Mathematician and his Task," begins by discussing the meaning of theories in physics and the role of mathematics in the development of these theories. Chapter 2, "The Tools of the Trade," gives special attention to complex numbers and to the development of the calculus and the differential equations of mathematical physics. The approach is essentially that of the physicist ("an infinitesimal quantity [is] one which does not exceed the smallest change of which we can take cognizance in our calculations").

The remaining five chapters, entitled respectively, "Ballistics or Newtonian Dynamics in War," "An Essay on Waves," "The Mathematics of Flight," "Statistics or the Weighing of Evidence," and "Mathematics and the Weather," are essentially independent essays that not only provide illustrations of applied mathematics in

action, but also serve to emphasize fundamentals in both classical and modern physics. An especially notable example is the approach to Heisenberg's uncertainty principle through the problem of the bandwidth required to resolve pulses of short duration. Perhaps the most stimulating chapter is the last, in which the author concludes that "Certain apparently sensible questions, such as the question of weather conditions . . . several days ahead, are in principle unanswerable and the most we can hope to do is to determine the relative probabilities of different outcomes."

There may be some question as to whether the non-mathematical layman will be able to follow all of the development of the last five chapters, and the author is occasionally guilty of extravagance. (Few aerodynamicists, even in the United Kingdom, would be willing to admit that F. W. Lanchester's esoteric volumes *Aerodynamics* and *Aerodonetics* "played a part in aerodynamics not unlike that exercised by Newton's *Principia* in astronomy.") These things notwithstanding, the reviewer believes that the author has succeeded admirably in reaching the goal described in the opening quotation of this review. Indeed, he goes beyond this goal, and the book (especially the individual essays) is warmly recommended to practicing applied mathematicians, as well as to laymen and students.

JOHN W. MILES

University of California
Los Angeles, California

16[K, S, W].—C. WEST CHURCHMAN & PHILBURN RATOOSH, Editors, *Measurement: Definitions and Theories*, John Wiley & Sons, Inc., New York, 1959, vii + 274 p., 24 cm. Price $7.95.

The December 1958 meetings of the American Association for the Advancement of Science included a two-day symposium on Measurement. The dozen papers of this symposium are included in this volume, together with a related paper separately invited but not delivered.

The book is divided into four parts. Part I, "Some Meanings of Measurement," consists of four papers concerned primarily with defining and characterizing the concept of measurement. Part II, "Some Theories of Measurement," contains three papers that approach the definition and characterization of measurement in more formal language—mathematical and logical symbols are more in evidence here—and thus warrant grouping together in a separate class. Part III, "Some Problems in the Physical Sciences," contains three papers that deal with theoretical and practical aspects of measurement in classical and modern physics, plus a paper on "rare events" that is out of place in this volume. Part IV, "Some Problems in Social Science," contains only two papers, one on inconsistency of judgments as a measure of psychological distance and one having to do with experimental tests of a probabilistic theory of economic behavior.

There are in all fourteen contributors, one paper being co-authored. The disciplines they represent are: philosophy (5), psychology (3), psychophysics (1), physics (2), mathematics (1), statistics (1), economics (1), and accounting (1). Philosophy is thus somewhat over-represented; astronomy, the biological and earth sciences, not at all.

In the Preface it is stated that the Symposium "was designed to present contrasts in approaches to the problems of measurement." To this end, "the partici-

pants were chosen from different disciplines" and selected particularly "because it was known that they had different viewpoints on the meaning and significance of measurement." Consequently, this volume "is a book of contrasts," not a unified collection of interrelated essays on measurement, and definitely not a textbook on measurement. At best it presents a broad picture of current thinking on the definition, nature, and functions of measurement, against a background of measurement needs and practices in various disciplines at the middle of the twentieth century. But the "picture" is not all in sharp focus. This is due not so much to differences in expository skill of the authors, as to differences in their objectives. Some seek sharpness of definition at the price of a narrow field of applicability; others demand a broad field of applicability at the sacrifice, if necessary, of sharpness of definition. Furthermore, much of the discussion is at a level of abstraction so far removed from the day-to-day practice of measurement in scientific and industrial laboratories that many who have devoted their lives to measurement of the properties of animate and inanimate things will find a large fraction of the volume very foreign to them, if not entirely unintelligible. Nevertheless, it is a volume that one will expect to find in the library of a university or college where research is conducted at the postgraduate level.

CHURCHILL EISENHART

National Bureau of Standards
Washington 25, D. C.

17[K, Z].—JAPANESE STANDARDS ASSOCIATION, *Random Number Generating Icosahedral Dice (20-face Dice)*, 6-1 Ginza-higashi, Chuo-ku, Tokyo. Price $2.50 per set of 3 dice + postage $.70 (up to 9 sets).

This device is a set of three icosahedral dice made of plastic material. The dice are different colors, red, yellow and blue, so that ordered triplets of digits may be generated. Each decimal digit appears on two faces of each die.

The dice were presumably intended to measure 15 millimeters between parallel faces. However, the casting was not particularly good and the measurements listed below were recorded between the ten pairs of faces on the new dice tested. This review was unaccountably lost for several months, and in the intervening period there has been considerable flow of the plastic material so that the measurements are currently considerably worse and actually meaningless, for the faces are clearly no longer plane.

Red	Yellow	Blue
14.74	14.90	15.09
14.97	14.96	14.71
14.82	14.95	14.93
14.88	14.97	14.93
14.94	15.05	14.93
14.76	14.95	14.83
14.85	15.07	15.09
14.96	14.97	14.74
14.83	14.93	14.87
14.79	14.90	15.03

When the dice were new they were tested by 800 rolls each on a level felt surface conforming to the specification of ordinary dice tables. Standard tests [1, 2] applied

to the individual dice and to the group of three detected no bias. Therefore, it seems reasonable to assume that the variations from a true regular icosahedron are minor in terms of the application intended.

Less meticulous samplings have been made with the aged dice, but it seems unlikely that a person generating random decimal digits at a rate which can be met by these dice would notice any serious bias.

The frequencies with which the digits appear can be changed slightly by the usual standard means. These would include weighting to displace the center of gravity (an awkward and cumbersome method at best, and, at worst, one which is difficult to disguise) or applying wax to one or more faces to increase the probability that the waxed face will be on the bottom after the throw.

C. B. TOMPKINS

University of California
Los Angeles 24, California

1. THE RAND CORP., *One Million Random Digits and 100,000 Normal Deviates*, The Free Press, Glencoe, Illinois, 1955.
2. C. B. TOMPKINS, RMT 11, *MTAC*, v. 10, 1956, p. 39–43.

18[L].—F. M. HENDERSON, *Elliptic Functions with Complex Arguments*, The University of Michigan Press, Ann Arbor, 1960, v + 38 p. + 160 unnumbered p., 29 cm. Price $8.00.

The tables and charts on the 160 unnumbered pages are in four equal parts, relating in turn to the functions $sn\ w$, $cn\ w$, $dn\ w$, and $E(w) = \int_0^w dn^2 w\, dw$, each for 19 values of the modular angle $\sin^{-1} k$, namely, 1°, 5°(5°)85°, 89°. Each opening has a table on the left and a corresponding chart on the right. If $w = u + iv$, the tables are all for $u/K = 0(.1)1$, $v/K' = 0(.1)1$, and the charts cover the same unit square on a scale such that the side of the square is about 16.2 cm. In Parts I–III, the quantities tabulated, to 4 figures without differences, are the real and imaginary parts x, y of $sn\ w$, $cn\ w$, $dn\ w$, respectively. In Part IV, if $E_R + iE_I = E(w)$, the relations $E(K) = E$, $E(K + iK') = E + i(K' - E')$ have led to the tabulation of the normalized quantities $E_R' = E_R/E$, $E_I' = E_I/(K' - E')$. The lines drawn on the charts are curves of constant x or y in Parts I–III, and curves of constant E_R' or E_I' in Part IV. Seven-figure values of the complete integrals are provided. The information given is sufficient to enable the four functions concerned to be evaluated for any point w in the complex plane.

The well-known tables of Spenceley and Spenceley [1] were used as a source of values for real w, whence the imaginary transformation and the addition formulas were used to compute the values of the functions of $u + iv$. The computation was mostly done on an IBM 650.

The Introduction contains enough information about elliptic integrals and functions to explain the tables and charts to anyone not previously acquainted with the subject. It also contains several applications to potential problems. It is pleasant to find such a valuable contribution to mathematical tabulation made by a civil engineer. The charts were constructed by a group of five Turkish naval officers at the University of Michigan.

One could wish that italic type (available and used in other contexts) had been used for mathematical symbols in the Introduction, but no such minor matter

can obscure the importance of the volume, which cannot fail to be found very useful. Only in the case of Part IV has the reviewer heard of any similar or related table of comparable scope, namely the table of the Jacobian zeta function by Fox and McNamee [2].

A. F.

1. G. W. Spenceley & R. M. Spenceley, *Smithsonian Elliptic Functions Tables*, Washington, 1947. See *MTAC*, v. 3, 1948, p. 89–92.

2. E. N. Fox & J. McNamee, "The two-dimensional problem of seepage into a cofferdam," *Phil. Mag.*, s. 7, v. 39, 1948, p. 165–203. See *MTAC*, v. 3, 1948, p. 246, 252, also v. 7, 1953, p. 190.

19[L].—L. Lewin, *Dilogarithms and Associated Functions*, MacDonald & Co. Ltd., London, 1958, xvi + 353 p., 21 cm. Price 65 Shillings.

The functions treated here are for the most part special cases of the Lerch zeta function, which can be defined by the series $\sum_{n=0}^{\infty} z^n/(n+b)^s$, $|z| < 1$, b not a negative integer or zero. To describe the text and tables, it is convenient to give some notation. Let $z = x + iy$, where x and y are real and $i = \sqrt{-1}$. Then

$$(1) \qquad Li_2(z) = -\int_0^z t^{-1} \ln(1-t)\, dt, \qquad Li_n(z) = \int_0^z t^{-1} Li_{n-1}(t)\, dt;$$

$$(2) \qquad Ti_2(x,a) = \int_0^x (t+a)^{-1} \arctan t\, dt, \qquad Ti_2(x) \equiv Ti_2(x,0),$$
$$Li_n(iy) = 2^{-n} Li_n(-y^2) + iTi_n(y);$$

$$(3) \qquad Cl_2(\theta) = -\int_0^\theta \ln\left(2\sin\frac{1}{2}t\right) dt, \qquad Cl_{2n}(\theta) = \int_0^\theta Cl_{2n-1}(t)\, dt,$$
$$Cl_{2n+1}(\theta) = Li_{2n+1}(1) - \int_0^\theta Cl_{2n}(x)\, dt;$$

$$(4) \qquad Gl_{2n}(\theta) + iCl_{2n}(\theta) = \sum_{k=1}^{\infty} \frac{e^{ik\theta}}{k^{2n}}, \qquad Cl_{2n+1}(\theta) + iGl_{2n+1}(\theta) = \sum_{k=1}^{\infty} \frac{e^{ik\theta}}{k^{2n+1}}.$$

The function $Gl_n(\theta)$ is a polynomial in θ of degree n.

Chapter I deals with the dilogarithm function $Li_2(x)$. The function $Ti_2(x)$ is considered in Chapter II; $Ti_2(x, a)$ in Chapter III. $Cl_2(\theta)$, θ real and positive, is Clausen's integral, and is studied in Chapter IV. Chapters V and VI take up $Li_n(z)$ for $n = 2$ and 3, respectively, and the analysis of this function for general values of n is the subject of Chapter VII. The general relations in (3) and (4) are also studied in this chapter. Chapter VIII deals with series expansions and integrals which can be expressed in terms of the basic functions in (1)–(4). Chapter IX is very useful. It is a compendium of results derived in the previous chapters. It also contains a survey of mathematical tables.

A description of the tabular material in this volume follows.

Table I.	$Li_n(x)$,	$n = 2(1)5$	
		$x = 0(.01)1.0$,	5D
Table II.	$Ti_n(y)$,	$n = 2(1)5$	
		$y = 0(.01)1.0$,	5D

Table III. $Cl_n(\frac{1}{2}\pi\alpha)$, $n = 2(1)5$
$\alpha = 0(.01)2.0$, 5D

Table IV. $Gl_n(\frac{1}{2}\pi\alpha)$, $n = 2(1)5$
$\alpha = 0(.01)2.0$, 5D

Table V. $Li_2(r, \theta)$ = real part of $Li_2(z)$, $z = re^{i\theta}$.
$r = 0(.01)1.0$, $\theta = 0(5°)180°$, 6D.

Throughout Table V the symbol x should be replaced by r. No information is supplied for interpolating in the tables.

The volume is replete with striking and curious results, some of which have been rediscovered a number of times, and the book should prevent future duplication of effort. There is a well-detailed table of contents and index. An extensive bibliography is also given.

Y.L.L.

20[L].—VERA I. PAGUROVA, *Tablitsy integro-eksponentsial'noĭ funktsii*

$$E_\nu(x) = \int_1^\infty e^{-xu} u^{-\nu} du$$

(*Tables of the Exponential Integral Function* $E_\nu(x) = \int_1^\infty e^{-xu} u^{-\nu}\, du$), Akad. Nauk SSSR, Vychislitel'nyy Tsentr, Moscow, 1959, xii + 152 p., 27 cm. Price 9.60 rubles.

This volume from the Computational Center of the Academy of Sciences of the USSR deals with well-known integrals which depend on the exponential integral when ν is a positive integer n. There are three tables. Table I (pages 3–52) is reproduced, with acknowledgment, from the NBS table calculated for a report of 1946 by G. Placzek and Gertrude Blanch (see *MTAC*, v. 2, 1947, p. 272) and more widely disseminated in 1954 in [1]. The table gives $E_n(x)$ to 7 or more decimals for $n = 0(1)20$, $x = 0(.01)2(.1)10$, and also 7-decimal values of the auxiliary functions $E_2(x) - x \ln x$ and $E_3(x) + \frac{1}{2}x^2 \ln x$ for $x = 0(.01).5$ and $x = 0(.01).1$, respectively; these last two ranges need transposing in the sub-title on page 1.

The other two tables are original. It is not stated what machines were used in computing them. Table II (pages 54–62) gives $e^x E_n(x)$, $n = 2(1)10$ to 7 decimals (6 figures) and e^x to 7 figures, all for $x = 10(.1)20$. Table III (pages 64–151) gives $e^x E_\nu(x)$, $\nu = 0(.1)1$ to 6 or 7 figures and e^x to 7 figures, all for $x = .01(.01)7(.05)12(.1)20$. No table gives differences.

A short introduction contains mathematical formulas and recommendations about interpolation. For integral n, the formula $d^r E_n(x)/dx^r = (-1)^r E_{n-r}(x)$ enables the tabulated function values themselves to be used for interpolation by means of Taylor's series. A table is given showing the accuracy attainable in interpolating various functions linearly or with 3 or 4 Taylor terms or with 3, 4, or 5 Lagrange terms.

A. F.

1. NAT. BUR. STANDARDS, Appl. Math. Ser. No. 37, *Tables of Functions and of Zeroes of Functions*, U. S. Government Printing Office, Washington, D. C., 1954, p. 57–111. See RMT 104, *MTAC*, v. 10, 1956, p. 249.

21[L].—Earl D. Rainville, *Special Functions*, The Macmillan Co., New York, 1960, xii + 365 p., 24 cm. Price $11.75.

This aptly titled, interesting, extremely well written book is based upon the lectures on Special Functions given by the author at the University of Michigan since 1946. The author's aim in writing the book was to facilitate the teaching of courses on the subject elsewhere. As an instructor in such a course, the reviewer feels certain that this most welcome text is to be accorded a warm reception on many college campuses.

More than fifty special functions receive varying degrees of attention; but, for the sake of usefulness, the subject is not approached on the encyclopedic level. Many of the standard concepts and methods which are useful in the detailed study of special functions are included. There is a great deal of emphasis on one of the author's favorite subjects: generating functions. Two interesting innovations are I. M. Sheffer's classification of polynomial sets and Sister M. Celine Fasenmyer's technique for obtaining recurrence relations for sets of polynomials. Functions of the hypergeometric family hold the center of the stage throughout a major portion of the text. The book concludes with a short current bibliography which should enable the reader to begin a more detailed study of the field.

There are twenty-one chapters in all, and the book may be roughly divided into four distinct parts:

(1) Two short preliminary chapters, 1 and 3, deal separately with infinite products and asymptotic series, respectively.

(2) Chapter 2 treats the gamma and beta functions, and chapters 4, 5, 6, and 7 are devoted to the hypergeometric family: the hypergeometric function, generalized hypergeometric functions, Bessel functions, and the confluent hypergeometric function, respectively.

(3) Chapter 8 is concerned with the generating function concept, as a preparation to chapters 9, 10, 11, and 12, which consider orthogonal, Legendre, Hermite, and Laguerre polynomials, respectively. Chapter 13 contains I. M. Sheffer's classification of polynomial sets; chapter 14 contains Sister M. Celine Fasenmyer's technique for obtaining recurrence relations for polynomials; and chapter 15 contains symbolic relations among classical polynomials. There follow three polynomial chapters: 16, 17, and 18, on Jacobi, ultraspherical and Gegenbauer, and other polynomials, respectively.

(4) The concluding three chapters, 19, 20, and 21, are devoted to elliptic functions, theta functions, and Jacobian elliptic functions, respectively.

J. B. Diaz

University of Maryland
College Park, Maryland

22[L].—L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge University Press, New York, 1960, x + 247 p., 29 cm. Price $12.50.

This is a valuable treatise on the subject, and the first of its kind in English. Tricomi [1] and Buchholz [2] have previously written books on the subject in Italian and German, respectively. Tricomi used the notation derived from the theory of hypergeometric functions, and this also is the principal notation employed in the Bateman Manuscript Project [3]. Buchholz uses the notation introduced by Whit-

taker. The notation of the present volume is a fusion of the two, and many results of the same character are given in all of the notations. This should prove most useful to research workers, as the confluent function has many applications, and notation is not uniform.

Chapter I studies the confluent hypergeometric functions as solutions of differential equations. Power series expansions are developed and relations between the various functions are carefully detailed.

Chapter II deals with differential properties including contiguous relations, Wronskians, addition theorems and multiplication theorems.

Chapter III is principally concerned with definite integrals involving the confluent functions. These include integrals of Barnes, Euler and Pochhammer types, and Laplace, Mellin and Hankel transforms.

Chapter IV takes up asymptotic expansions. It is a very useful compendium. In particular, as the author remarks, the theory is not complete, and it is in this area where important new results are anticipated.

Related functions such as Coulomb wave functions, Bessel functions, incomplete gamma functions, etc., are briefly considered in Chapter V.

Descriptive properties are the subject of Chapter VI. Zeros in x of ${}_1F_1(a; b; x)$ and $W_{k,m}(x)$, and zeros in a and b of the former are studied. Formulas and expansions for the zeros are provided. Numerical evaluation of ${}_1F_1$ is discussed and numerous figures are provided to illustrate its behavior when a, b, and x are real.

Three appendices give tables. These are as follows:

Appendix I. Smallest positive zeros of ${}_1F_1(a; b; x)$

$a = -4.0(0.1) - 0.1, \quad b = 0.1(0.1)2.5, \quad$ 7D

Appendix II. ${}_1F_1(a; b; x)$

$a = -1.0(0.1)1.0, \quad b = 0.1(0.1)1.0, \quad x = 0.1(0.1)10.0,$

7S – 9S

Appendix III. ${}_1F_1(a; b; x)$

$a = -11.0(0.2)2.0, \quad b = -4.0(0.2)1.0, \quad x = 1, \quad$ 7D

Appendix I was calculated to 8D on EDSAC, and the error does not exceed two units in the seventh decimal. Appendix II was calculated on the same machine, but no mention is made of accuracy. Some spot checks indicate that the error does not exceed four units in the last figure recorded. Appendices I and II are extensions of tables previously published by the author. Other short tables of the material in Appendix II have also appeared previously, but the present table is the most complete available. Appendix III was evolved using recurrence formulas and the data of Appendix II. It is quite surprising that nothing is said about interpolation.

A detailed table of contents, general index, and symbolic index of notation enhance the usefulness of the volume. The list of references, though not exhaustive, is fairly complete for work subsequent to that of Buchholz [2], whom the reader should consult for an extensive bibliography of work prior to about 1952. The reviewer notes that two important tables of Whittaker functions reviewed in [4] are not found in Miss Slater's list of references.

Y. L. L.

1. F. G. TRICOMI, *Lezioni sulle funzioni ipergeometriche confluenti*, Gheroni, Torino, 1952.
2. H. BUCHHOLZ, *Die konfluente hypergeometrische Funktion mit besonderer Berücksichtigung ihrer Anwendungen*, Springer, Berlin, 1953.
3. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, & F. G. TRICOMI, *Higher Transcendental Functions*, v. 1, Chapter VI, McGraw-Hill, New York, 1953.
4. RMT 46, *MTAC*, v. 12, 1958, pp. 86–88.

23[M, X].—G. DOETSCH, *Einfuhrung in Theorie und Anwendung der Laplace-Transformation*, Birkhäuser Verlag, Basel, Switzerland, 1958, 301 p., 24 cm. Price SFr 39.40.

This book forms an excellent introduction to the subject of Laplace transformations in one dimension, written by one of the leading experts in the field. From his wide knowledge of both the theoretical and applied aspects of the subject, the author has written a very readable, rigorous exposition which also indicates relations with appropriate physical concepts.

After a general introduction, the basic properties of the Laplace transform are developed in ten short chapters. Included are discussions of half-planes of convergence, uniqueness of the inverse, analytic properties of the transform, the effect of a linear transformation of the independent variable on the transform, the effect of differentiation and integration, and the transformation of a convolution. Because each chapter is devoted to a separate topic, the book is very useful for reference purposes. Many examples of specific transforms are given.

The next four chapters deal with the application of Laplace transforms to the following problems: the initial-value problem in ordinary differential equations with constant coefficients; the solution of differential equations for special input functions; homogeneous and non-homogeneous systems of differential equations; the initial-value problem for difference equations.

The next group of chapters deals with further properties of the transform, some of which may not be familiar to the reader: the behavior of the transform at infinity; inversion formulas expressed as integrals along vertical lines, as integrals along deformed paths in the complex plane, and as series of residues; conditions for the representation of a function as a transform; functions given as the sums of series of transforms; the analogue of Parseval's formula and transforms of products; and the asymptotic behavior of the transform and of the "original" function.

The book concludes with three chapters on further applications of the Laplace transform to differential equations with variable coefficients, to simple partial differential equations, and to certain integral equations.

A useful feature of the book is the inclusion of necessary background material, particularly in the later chapters.

DOROTHY L. BERNSTEIN

Goucher College
Towson, Baltimore 4, Maryland

24[S, X].—G. I. MARCHUK, *Numerical Methods for Nuclear Reactor Calculations*, (An English translation of a work originally published in Russian as Supplement Nos. 3–4 of the Soviet Journal of Atomic Energy, Atomnaîa Énergiîa. Atomic Press, Moscow, 1958.) Consultants Bureau, Inc., New York, 1959, 293 p. 28 cm. Price $60.00.

A more accurate title for this book would have been *Numerical Methods for*

Nuclear Reactor Physics Calculations. As stated by the author, "the book is an attempt at a more or less systematic exposition of numerical methods for the calculation of thermal, intermediate, and fast neutron reactors. Particular attention is devoted to the problems of critical mass, the space-energy neutron flux distribution, and the neutron importance." This is the first book to become available in English that is entirely devoted to this area of calculation, and as such is a helpful addition to our literature.

The book is clearly not addressed to mathematicians. A familiarity with nuclear reactor theory is assumed; terms such as "cross section" are used without definition. The mathematician will probably be disturbed by such statements as "the integrands in the resulting two terms are now continuous, and they can therefore be expanded in a Taylor's series about r_k" (p. 106), or, "primarily, this is due to the fact that a second-order difference equation has two eigenvalues, one of which is positive" (p. 122). Emphasis throughout the book is on the derivation of equations and the mechanics of their solution. The stability of a number of the numerical methods against the exponential growth of round-off errors is considered. Problems related to the existence of solutions and the accuracy with which solutions of the various approximate equations represent the desired solutions are mentioned only occasionally.

In the foreword the author mentions that the numerical methods were "first tested on a large quantity of theoretical and experimental data and were later used in operation." It is unfortunate that the results of some of these calculations were not included in the book. Results of specific calculations are not given. For example, in Chapter VIII, various sets of finite-difference equations are set up to approximate the diffusion equation. Some of these difference equations are more accurate than others. However, no discussion of the actual errors incurred when these various approximations are used is presented other than a statement, "with a sufficiently fine network good accuracy can be obtained." This would have been an excellent place to include numerical results showing the kind of accuracy that could be attained when the various approximate equations are used in typical specific problems.

A useful feature of this book is that corresponding to every set of equations derived, an adjoint set of equations is also obtained.

In the first chapter a basic set of transport equations is derived: a "slowing-down" equation in which the assumption is made that the moderator nuclei are at rest while the neutrons are slowed down by elastic collisions, and a "thermal" equation in which the velocities of the moderator nuclei are assumed to have a Maxwellian distribution. In succeeding chapters, various sets of equations approximating this basic set are obtained. In the second chapter the diffusion approximation is derived following the method of R. E. Marshak, H. Brooks, and H. Hurwitz, Jr., published in *Nucleonics*, v. 4, 5 (1949), in which the angular dependence of the neutron flux and the collision function are both approximated by retaining the first two terms in a series expansion involving spherical functions. The corresponding boundary conditions are also derived. In the third chapter the diffusion-age approximation is obtained when the energy dependence of the flux is approximated by retaining the first few terms of a Taylor series expansion. Here again, the corresponding boundary conditions are derived. In the fourth chapter further approximations to the basic transport equation are derived which take more accurate account of energy dependence than the diffusion-age approximation,

and are therefore applicable in cases where there is stronger absorption of slowing-down neutrons. Equations are derived in which resonance absorption is accounted for based on a method suggested by Wigner. Also, the Greuling-Goertzel equations for the slowing-down of neutrons in an inhomogeneous medium are obtained.

In Chapter V exact solutions are given of the basic and adjoint equations of the diffusion-age approximation for a homogeneous, unreflected reactor with an extrapolated boundary which is energy-independent. Also, for this special case the convergence of "the method of successive approximations" for finding the criticality factor is established. "The method of successive approximations" means here that at each step, assuming an estimate of the fission source, the corresponding neutron flux is calculated, and then using this set of flux values, a new estimate of the fission source is obtained. The criticality factor is obtained using ratios of successive source estimates.

In Chapter VI various sets of multigroup equations are derived which are applicable primarily to thermal reactors where weak absorption of slowing-down neutrons is assumed; in Chapter VII various sets of multigroup equations are derived which are applicable primarily to intermediate reactors where strong absorption of slowing-down neutrons must be considered.

Chapter VIII contains the derivation of various sets of three-point difference equations to approximate the one-dimensional diffusion equation in plane, cylindrical, and spherical geometries. In addition, five-point difference equations are obtained for the two-dimensional diffusion equation in plane (x, y) and cylindrical (r, z) geometries.

In Chapter IX various methods are discussed for solving sets of finite-difference diffusion equations of the types set up in the previous chapter. To solve the three-point difference equation a well-known factorization technique is described, and its stability against the growth of round-off errors is considered. For the solution of the five-point difference equations an analogous factorization technique, as well as a simple iteration procedure, and a third method suggested by N. I. Buleev are considered. Error criteria which can be used to estimate the accuracy of results are also mentioned.

Chapter X contains a discussion of the "net" method for obtaining approximate solutions of the differential slowing-down equation in one spatial dimension. Difference equations are set up to approximate the differential equation and solved for the unknown function in one energy interval in terms of values computed for the preceding interval. Both explicit and implicit schemes are discussed. Their stability is investigated by applying the von Neumann stability criterion.

In Chapter XI perturbation theory is applied to obtain approximate formulas giving the change in the criticality factor when small changes are made in values of the various physical parameters. This is done starting with the basic equations of Chapter I as well as with the various equations obtained in later chapters to approximate these basic equations. Some further applications of perturbation theory are discussed such as the calculation of the reactivity equivalent of a control rod in the center of the core of a cylindrical reactor, as well as for a whole system of control rods in such a reactor.

Chapter XII on heterogeneous effects in nuclear reactors contains descriptions of the Gurevich-Pomeranchuk theory of resonance absorption applicable in calculating the effective resonance integral for "thin" U^{238} blocks, the Wigner theory for

calculating the resonance integral for "thick" blocks, and Orlov's extension of Wigner's method giving a formula for the effective resonance integral which can be used for U^{238} blocks of any thickness. Further, a method is given which was developed by Marchuk and Orlov for calculating the resonance capture of neutrons in a plane lattice of uranium blocks. Sedel'nikov's extension of the Gurevich-Pomeranchuk theory to take account of self-shielding in intermediate neutron reactors is described. Effective boundary conditions for "black" and "grey" bodies are considered. Finally, the Wigner-Seitz method is applied to a cell of a heterogeneous reactor to obtain effective constants for an equivalent homogeneous reactor.

Chapter XIII contains a discussion of the calculation of the neutron flux in fast nuclear reactors. The transport equation including terms for inelastic scattering is given. Corresponding multigroup equations are obtained. For solving these equations both the method of spherical harmonics and Carlson's S_n method are described.

A final chapter on calculations for intermediate and thermal reactors with hydrogenous moderators is included. This chapter was added after the original manuscript had been prepared for publication.

It is unfortunate that this book was not more carefully edited. For example, there is confusion in the use of the terms 'flux' and 'current'; on page 104 the following statement appears, "we wish to find a solution of (30.1) which has a continuous flux

$$I = r^n D \frac{d\phi}{dr}."$$

Throughout the text, the equations which are usually referred to as "transport" equations are called "kinetic" equations. Too frequently proper names are spelled phonetically as a result of the transliteration process, rather than with their usual English spelling; for example, R. E. Marshak appears as R. E. Marchak (p. 6), R. Ehrlich appears as R. Erlich (p. 6), Neumann appears as Neiman (p. 142), etc. Also, it is unfortunate that the publishers did not take more care in their representation of symbols. The equations were apparently reproduced photographically. A parameter represented by a script letter in an equation is frequently represented in another form in the text. On pages 132 and 134 a parameter which appears in the equations as a "chi" is represented as a "kappa" in the text.

This book is a very useful addition to our literature. It is hoped that its presence and imperfections will act as a stimulus for the publication of another book in the area of numerical methods for nuclear reactor calculations which will give a more satisfying mathematical treatment of this subject, and which will be made available at a more reasonable price.

E. CUTHILL

Applied Mathematics Laboratory
David Taylor Model Basin
Washington 7, District of Columbia

25[W].—E. BURGER, *Einfuhrung in die Theorie der Spiele*, Walter de Gruyter & Co., Berlin, 1959, 169 p., 23 cm. Price DM 28.

This is the first book on the theory of games to appear in the German language. The author, professor of mathematics at the University of Frankfurt, has previously

given several courses on the subject for economists, but the emphasis in this book is to bring out the mathematical structure of the theory. He has succeeded in this in an admirable way. The comparatively few applications are meant as illustrations rather than as developments of the empirical fields for which the theory of games is particularly suited. There is an extensive discussion of zero-sum two-person games, and of equilibrium points for non-cooperative games. The presentation of linear programming is also clear and rather exhaustive. The theory of cooperative games, that is, those in which the formation of coalitions is advantageous and allowable, and in which side-payments by the players are freely admitted, is developed in fair detail, even including a discussion of Shapley's value of an n-person game. At this point the author confesses that he is less sure of the intuitive background against which this theory has been placed, a position that is common to many mathematicians who have studied the problem of n-person games. However, this is a difficult issue and the author is wise not to have taken too definite a stand, rather withdrawing to the strictly mathematical aspects involved. These problems can only be solved by recourse to an improved description of the socio-economic world. If it turns out that the real problem involves a high degree of cooperation—and I have no doubts whatsoever that this will be the case—then the mathematical theory will have to accommodate itself to these facts, even if the mathematical structure is uncommon and cumbersome, until fundamentally new concepts are established.

Dr. Burger possesses a very high didactic skill; the great clarity which pervades his whole book should make it a welcome tool for the novice in game theory who commands the mathematical knowledge expected of first or second year graduate students. A translation of the work should be seriously considered, since there is no similar book in English which accomplishes as much in such small compass as Dr. Burger's does.

OSKAR MORGENSTERN

Princeton University
Princeton, N. J.

26[X].—R. P. BOAS, JR. & R. C. BUCK, *Polynomial Expansions of Analytic Functions*, Springer-Verlag, Berlin, 1958, viii + 77 p., 23 cm. Price DM 19.80.

A great part of numerical analysis is concerned with polynomial approximation to analytic functions, and so this booklet appears of immediate interest to the numerical analyst. However, with numerical analysis in its present state, it is more relevant for studies in the general theory of functions of a complex variable or in the theory of special functions.

Given a set of polynomials $\{p_n(z)\}$ and a function $f(z)$, it is reasonable to ask whether we can find coefficients $\{c_n\}$ such that

$$f(z) = \sum c_n p_n(z) \tag{1}$$

in some sense. This is the "expansion problem".

It is also reasonable to ask whether, given linear functionals $\{L_n(f)\}$, for example,

$$L_n(f) = f^{(n)}(0) \tag{2}$$

or

$$L_{2n}(f) = f^{(2n)}(0), \qquad L_{2n+1}(f) = f^{(2n)}(1), \tag{3}$$

we can obtain a representation of f in the form (1). This is the "interpolation problem". The solution in case (2) is given by the Taylor expansion and that in (3) by the two-point Lidstone expansion

$$f(z) = \sum A_n(1 - z)f^{(2n)}(0) + \sum A_n(z)f^{(2n)}(1),$$

where the polynomials $A_n(z)$ are defined by

$$\frac{\sinh zw}{\sinh w} = \sum A_n(z)w^{2n}.$$

This is valid, for instance, when $f(z)$ is an entire function of exponential type less than π.

These two problems are studied by a general method—"kernel expansion"—for wide classes of polynomials (defined by generating functions) and for the cases when $f(z)$ is entire, or regular at the origin.

The material is accessible to those familiar with the classical methods of complex variable theory, and its study by numerical analysts is recommended. It will, for instance, encourage us to get off the real axis, reveal some thought-provoking "bad examples" (e.g., non-uniqueness in (1)), and show us how mathematics should be written.

JOHN TODD

California Institute of Technology
Pasadena, California

27[X].—RUDOLPH E. LANGER, Editor, *Symposium on Numerical Approximation*, The University of Wisconsin Press, Madison, 1959, x + 462 p., 24 cm. Price $4.50.

The present volume auspiciously launches the publication efforts of Professor Langer's group at the Mathematics Research Center. It contains the proceedings of a symposium conducted at the University of Wisconsin, April 21–23, 1958. The papers (original and expository) present such a well-rounded survey of numerical approximation because of the careful selection of the invited contributors. A listing of the authors and the titles of their papers will serve to indicate the level and scope of the work.

A. M. Ostrowski	On Trends and Problems in Numerical Approximation
R. C. Buck	Linear Spaces and Approximation Theory
Z. Kopal	Operational Methods in Numerical Analysis Based on Rational Approximations
P. J. Davis	On the Numerical Integration of Periodic Analytic Functions
H. E. Salzer	Some New Divided Difference Algorithms for Two Variables
P. C. Hammer	Numerical Evaluation of Multiple Integrals
M. Golomb	Optimal Approximation and Error Bounds
A. Sard	The Rationale of Approximation
J. L. Walsh	On Extremal Approximations
E. L. Stiefel	Numerical Methods of Tchebycheff Approximation
L. Fox	Minimax Methods in Table Construction

T. S. Motzkin	Existence of Essentially Nonlinear Families Suitable for Oscillatory Approximation
I. J. Schoenberg	On Variation Diminishing Approximation Methods
M. Golomb	Approximation by Functions of Fewer Variables
J. C. P. Miller	Extremal Approximations—A Summary
R. C. Buck	Survey of Recent Russian Literature on Approximation
F. L. Bauer	The Quotient-Difference and Epsilon Algorithms
J. B. Rosser	Some Sufficient Conditions for the Existence of an Asymptotic Formula or an Asymptotic Expansion
J. W. Tukey	The Estimation of (Power) Spectra and Related Quantities
L. Collatz	Approximation in Partial Differential Equations
J. Todd	Special Polynomials in Numerical Analysis

E. I.

28[X].—RUDOLPH E. LANGER, Editor, *Frontiers of Numerical Mathematics*, The University of Wisconsin Press, Madison, 1960, xi + 132 p., 24 cm. Price $3.50.

This book contains eight papers and a discussion of these papers presented at a Symposium held at Madison, Wisconsin on October 30–31, 1959. This symposium was conducted jointly by the Mathematics Research Center, the United States Army, and the National Bureau of Standards. Its purpose "was not intended to be an occasion for the presentation of research results, but one for a survey of the future; for the identification of some mathematical problems that will have to be faced in the lines of scientific advance."

The authors and the titles of their papers are:

William Prager	Stress Analysis in the Plastic Range
Garrett Birkhoff	Some Mathematical Problems of Nuclear Reactor Theory
Zdenek Kopal	Numerical Problems of Contemporary Celestial Mechanics
Lee Arnold	Aeroelasticity
Phillip M. Morse	Operations Research
Joseph O. Hirschfelder	Mathematical Bottlenecks in Theoretical Chemistry
S. Chandrasekhar	Magnetohydrodynamics
J. Smagorinsky	On the Application of Numerical Methods to the Solution of Systems of Partial Differential Equations Arising in Meteorology

It is unfortunate that the paper by Lee Arnold was made from a tape recording and was not finally reviewed by the author. It is very difficult for a reader to follow this paper, for many statements in it seem to refer to illustrations which are not included.

The reader of this book should not expect to find a detailed discussion of numerically formulated problems arising in various branches of science. He will find, in the main, discussions and reviews of various open mathematical problems in different scientific areas. Some of the papers refer to numerical treatment of these problems. A surprising number indicate a strong preference on the author's part for analytical methods for dealing with their problems.

One can but agree with J. Smagorinsky when he states "that computing ma-

chines cannot be considered a substitute for the ingenious mathematical and laboratory techniques of analysis which have been devised. . . ." However, this reviewer is convinced that such ingenuity when properly coupled with the power of modern computers will provide greater insight than analytical methods alone.

A. H. T.

29[X].—(a) D. S. MITRINOVIĆ, "Sur les nombres de Stirling de première espèce et les polynômes de Stirling," *Publ. de la Fac. d'Électrotechnique de l'Univ. à Belgrade* (Série: Math. et Phys.), No. 23, 1959, 20 p. (Serbian with French summary. Tables by Miss RUŽICA S. MITRINOVIĆ.)

(b) D. S. MITRINOVIĆ & R. S. MITRINOVIĆ, "Sur les polynômes de Stirling," *Bull. Soc. Math. Phys. Serbie*, v. 10 (for 1958), p. 43–49, Belgrade. (Summary in Russian.)

(c) D. S. MITRINOVIĆ & R. S. MITRINOVIĆ, "Tableaux qui fournissent des polynômes de Stirling," *Publ. Fac. Élect. Univ. Belgrade* (Série: *Math. et Phys.*), No. 34, 1960, 24 p. (Summary in Serbian.)

These three papers are concerned with the Stirling numbers of the first kind, S_n^r, which may be defined for positive integral n by

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{r=0}^{n} S_n^r x^r.$$

Altogether the numbers S_n^{n-m} are tabulated for $m = 1(1)32$, $n = m + 1(1)N$, where $N = 200$ for $m = 1(1)5$, $N = 100$ for $m = 6$, and $N = 50$ for $m = 7(1)32$. The values for $m = 1(1)7$ are given in (a), for $m = 8(1)13$ partly in (a) and partly in (c), for $m = 14(1)20$ partly in (b) and partly in (c), and for $m = 21(1)32$ in (c). The authors found no discrepancy as a result of some checking against unpublished tables by F. L. Miksa (see *MTAC*, v. 10, 1956, p. 37).

Algebraic expressions for S_n^{n-m} in the form of binomial coefficients $\binom{n}{m+1}$ multiplied by polynomials (with factors $n(n-1)$ separated out if m is odd and not less than 3) are given for $m = 1(1)13$ in (a) and for $m = 1(1)9$ in (c).

S_n^{n-m} may also be expressed as a sum of multiples of binomial coefficients in the form

$$S_n^{n-m} = \sum_{k=0}^{m-1} C_m^k \binom{n}{2m-k}.$$

Altogether the values of the coefficients C_m^k are given for $k = 0(1)31$, $m = k + 1(1)32$, the values for $k = 0(1)19$, $m = k + 1(1)20$ being found in (b) and the remaining values in (c).

A. F.

30[Z].—WAYNE C. IRWIN, *Digital Computer Principles*, Van Nostrand Co., Inc., Princeton, 1960, vi + 321 p., 24 cm., $8.00.

This book contains material presented at a training course in the Electronics Division of the National Cash Register Company. It is an extremely elementary "book for the beginner. No previous acquaintance with computers, electronics or mathematics is necessary."

The book is divided into nine sections and forty-two chapters. The section titles are: Methods of Computation, Symbolic Logic, Mechanization of Logic, Mechanization of Storage, Timing, Mechanization of Arithmetic, Control, Communication with the Computer, Preparation of Instructions, Reduction of Errors, and Present Trends.

The author has placed more emphasis on symbolic logic than on any other subject. This section of the book begins with Boolean algebra and terminates with a discussion of the Harvard Minimizing Chart. However, even in this section, the best written one in the book, the author does not develop the mathematical ideas involved adequately or prove statements. He describes various facts and illustrates the use of various techniques.

The author does not go very far in the discussion of any topic he selects, and he omits many significant points in his discussion. Some omissions are noted in footnotes but some are never mentioned. For example, "asynchronous machines" are dismissed with a footnote on page 145. The words "round-off" and "rounded multiplication" are never mentioned in the text and neither word can be found in the index. The existence of a computer using a number representation other than one with a signed absolute value is never mentioned.

It is unfortunate that the author did not see fit to include in the bibliography references to the reports by von Neumann and his co-workers at the Institute for Advanced Study. The discussion given in these reports of arithmetic performed in a computer with a two's complement representation of numbers is clear and complete, and would supplement very nicely the author's limited discussion of arithmetic in a binary machine.

A. H. T.

31[Z].—T. E. IVALL, *Electronic Computers, Principles and Applications*, Second Edition, Philosophical Library, New York, 1960, viii + 263 p., 22 cm. Price $15.00.

This is a revised version of the first edition published in 1956 under the same title [1]. In the first edition the author acts as an editor, publishing in book form the separate contributions of a number of English writers working in the field of computers. In the present edition the author attempts to regroup and rewrite the material so as to present a more coherent picture. He also adds three short chapters, as follows:

(1) In place of one chapter on Analogue Computing Circuits the new edition now contains two chapters, Analogue Computing Circuits—1, and Analogue Computing Circuits—2.

(2) A chapter on Programming Digital Computers is added.

(3) The chapter, Computers of the Future, somewhat rewritten, now becomes Recent Developments, and a new chapter, Computers of the Future, is added.

The book is intended as an introduction to the computer field for the non-technical reader. It can adequately fulfill this purpose. However, the book has a number of deficiencies. It also appears that the author has not really kept abreast with modern developments in this field, nor with the rapid advances which are taking place in this country. In his brief chapter, Computers of the Future, he discusses mainly some future potential applications of computers rather than the exciting developments which are taking place in the field of computer design. He sometimes betrays

a rather limited appreciation of the flexibility and power of modern digital computers. For instance, on page 214 he states: "It now appears that digital computers, too, may have some applications in the field of simulation, particularly when the system to be simulated has a digital nature." Digital computers have for years been used to simulate very complex physical systems, for example, the temperature distribution within the core of a nuclear reactor.

This book can serve as a simple and not too technical introduction to computers. In the opinion of the reviewer, however, the interested reader can find a number of other books recently published that will more adequately serve this purpose, and probably at a lower price.

H. P.

1. T. E. IVALL, Editor, *Electronic Computers*, Philosophical Library, New York, RMT 118, *MTAC*, v. 12, 1958, p. 255.

32[Z].—WALTER J. KARPLUS, *Analog Simulation, Solution of Field Problems*, McGraw-Hill, New York, 1958, xv + 434 p., 23 cm. Price $10.00.

Analog Simulation by Karplus is essentially unique. There are neither competitive contemporary treatises nor older works with which to compare it in review. It almost automatically, then, belongs in any complete library of machine computation. The real question, on the other hand, is whether a real need exists for either a text or a reference book on its subject matter. As a research engineer, the reviewer has a strong affirmative answer to the second half of this question. He can now throw away the collection of reprints and notes which he has been carrying around for years. Its merit as a text is more debatable.

The subject matter of the book should be made clear first. The title, *Analog Simulation*, conveys to most American engineers devices like aircraft and missile simulators and electronic amplifier differential analyzers. Even the subtitle, "Solution of Field Problems," evokes a momentary picture of a fire control computer in the "field." The full title is correctly descriptive of the category of physical problems considered, but a title like "Analog Techniques for Partial Differential Equations" would have made the material covered more readily apparent.

The book is divided into three major parts. Part 1 (Chapters 2–4) gives 98 pages of mathematical background for analog study of field problems. In Part 2 (Chapters 5–10), the actual "hardware" is discussed in 165 pages. Part 3 returns to a mathematically organized discussion of the applications of the previously discussed analog techniques to different classes of partial differential equations (Chapters 11–14, 115 pages). It may be seen that most of the book is problem-oriented.

In spite of this problem orientation, the book is highly recommended as a reference on analog techniques. This is in large part a consequence of the superb list of references at the end of each chapter and the excellent 22-page general bibliography. The only serious omission is *Dynamical Analogies* by H. F. Olson, the closest equivalent to Karplus, although almost solely concerned with acoustics and vibration. If for no other reason, it would be recommended for a place on the engineer's bookshelf alongside good source books on digital computers and electronic differential analyzers.

The reviewer feels that information on direct analog simulators should be more readily available to engineers. The practitioners of the computing art tend to develop rather strong prejudices for the techniques with which they are more familiar. In fact, the three major classes of computers (digital data processors, electronic differential analyzers, and direct analog devices) all figure with equal importance in current engineering research work for which the reviewer is responsible. Not only do we have the continual rather pointless argument about the relative merits of digital versus analog computation, but computer engineers working with operational amplifier machines tend to disparage passive-network and other direct analog devices. The Introduction (Chapter 1) and Chapter 9 on "Electronic Analog Computers" do an excellent job of putting this controversy into proper perspective. The analog systems discussed in Part 2 are not diminishing in importance to engineering. These are: Chapter 5, "Conductive-Solid Analogs"; Chapter 6, "Conductive Liquids—The Electrolytic Tank"; Chapter 7, "Resistance Networks"; Chapter 8, "Resistance-Reactance Network Analogs"; Chapter 10, "Nonelectric-analog Simulation Systems." The value of *Analog Simulation* as a unique reference on distributed system analogs is indisputable.

Some consideration must be given to the merits of the book as a student text, since it is intended for such by the author. The first proposed use for the second half of a one-semester upper-undergraduate or first-graduate introduction to analog computation does not appeal to the reviewer. At this level, I have strong prejudices in favor of orientation along the line of "mathematical and physical principles of engineering analysis," with all techniques of computation introduced for illustration and some familiarization, plus some other topics such as scale modeling and dimensional analysis. My principal reason for objecting to a course based on Karplus is that I feel mathematics beyond an engineering course in ordinary differential equations is required; probably at least an introduction to complex variables or the usual survey course in applied mathematics. As a text for a full-semester course for graduate students, however, it appears excellent. Part 1, "The Mathematical Model," would be an excellent "refresher" on the mathematical and physical principles involved. Graduate engineering students working in mechanical engineering areas should benefit by the familiarization with electrical engineering techniques.

The reviewer's major criticisms of the book are very few. The weakest part is Chapter 3, "Transformations." Although it is a reasonably adequate condensed summary with useful reference tables of transformations, as a text for someone with no previous introduction to the subject it would be useless. The caution on page 66 about applicability of conformal transformations in space, although quite correct, would only be mystifying after a four-page discussion of the subject. The only misprint of any consequence noted is also in this chapter, namely $F(t)$ for $f(t)$ in formula (3.23). There are also occasional implications that physical intuition can be substituted for mathematical understanding, as on pages 11, 12, where it is stated that "It is entirely possible to develop analogs without taking a mathematical route —namely, by applying direct physical insight." Again, on page 79, Karplus warns that use of finite-difference expansions requires an understanding of errors and approximations inherent in the process, and conditions for a valid solution then immediately suggest that physical reasoning and insight may be substituted for mathe-

matical understanding. To the reviewer, a contradiction is implied, which, it is doubtful, was intended. It is considered dangerous to suggest to students that physical intuition may be used in solving problems, without parallel mathematical analysis of the properties of the analogous systems involved. Otherwise, there is nothing that can be cited to detract from the value of *Analog Simulation* as a unique contribution to its field.

W. H. KEEN

Bureau of Naval Weapons
Navy Department
Washington, D. C.

33[Z].—W. OPPELT, Compiler, *Anwendung von Rechenmaschinen bei der Berechnung von Regelvorgängen (Application of Computing Machines to the Computation of Servomechanisms)*, R. Oldenbourg, Verlag, Munich, 1958, 128 p., 24 cm. Price DM 16.80.

This book consists of ten papers presented at a meeting in Düsseldorf on November 8, 1957. The authors and titles are:

1. E. Bucovics	On the Use of Small Digital Computers for the Solution of Fundamental Problems of Servomechanics
2. J. B. Reswick	A Simple Graphical Method for Deconvolution
3. A. Ieonhard	A Special Computer for Polynomials
4. R. Herschel	On the Design of Analog Circuits for Problems of Servomechanics
5. O. Foellinger & G. Schneider	Comparison of Computations for Servomechanisms using Computing Machines of Different Types
6. E. Buehler	On the Mechanical System with Friction and its Electronic Equivalent
7. H. Witsenhausen	Application of Analog Computers for Optimizing Discontinuous Control Circuits with Randomly Varying Input Parameters
8. D. Ernst	Practical Work with Analog Computers
9. Th. Stein	On the Usefulness of Analog Experiments in Practical Applications
10. W. Roth	Investigation of the Control Characteristics of Electric Generator Sets by the Use of Analog Computers

(All papers are in German, except the second.)

Only two papers consider digital computers: No. 1 describes in much detail the use of an IBM 604 calculating punch for some basic problems; No. 5 compares the accuracy and speed of two analog computers (one mechanical, one electronic) with that obtained on the IBM 650 in the BELL interpretative mode. The comparison is unfair, however, since the Runge-Kutta method is used for solving a system of linear differential equations with constant coefficients (time: 27 sec./step for a system of six equations!).

There is no space here for reviewing each paper separately, so that the following statements may do some injustice to individual papers. Most authors report mainly on their practical work and experiences, not on new theoretical results. There are a

fair number of references, but some are geographically biased and/or obsolete. The book gives an interesting picture of the local situation (Germany and Austria) in the mid-fifties; but as far as applications of computing machines, and in particular digital computers, are concerned, it was almost obsolete when it appeared in 1958 and is, of course, even more nearly obsolete today.

HANS J. MAEHLY

Syracuse University
Syracuse 10, N. Y.

TABLE ERRATA

296.—K. MITCHELL, "Tables of the function $\int_0^x \frac{-\log|1-y|}{y}\,dy$, with an account of some properties of this and related functions," *Phil. Mag.*, v. 40, 1949, p. 351–368.

On page 359, Table II: in column 4, first entry, *for* 0.05 6039 292 *read* 0.05 0639 292.

R. P. BRUMBACH, JR.

University of Notre Dame
Notre Dame, Indiana

297.—G. W. SPENCELEY, R. M. SPENCELEY, & E. R. EPPERSON, *Smithsonian Logarithmic Tables to Base e and Base 10*, The Smithsonian Institution, Washington, D. C., 1952.

The following errata supplement those recently reported by me (*Math. Comp.*, v. 14, p. 308, MTE **283**).

		For	Read
p. 5	$\log_e 245$	... 98481 147	... 98481 146
p. 10	$\log_e 472$	... 85553 ...	... 85555 ...
p. 176	$\log_e 8772$	9.07832 ...	9.07932 ...
p. 185	$\log_e 9215$	9.12856 ...	9.12858 ...
p. 287	$\log_{10} 4213$	63459 ...	62459 ...
p. 330	$\log_{10} 6381$	... 87448 ...	... 87446 ...

CHARLES R. SEXTON

2947 Elmwood Court
Berkeley 5, California

298.—I. M. VINOGRADOV, *Elements of Number Theory*, Translated from the fifth edition by Saul Kravetz. Dover Publications, Inc., New York, 1954.

The following typographical error has been found, supplementing errata previously reported [1]:

p. 227 *for* 5373, *read* 3373.

SIDNEY KRAVITZ

592 Herrick Drive
Dover, New Jersey

1. *MTAC*, v. 9, 1955, p. 125, RMT **59**; v. 12, 1958, p. 108, MTE **260**.

CORRIGENDA

HERBERT E. SALZER, "A note on the solution of quartic equations," *Math. Comp.* v. 14, 1960, p. 279–281.

p. 280, line 8, ...depending upon whether m is real ....
should read ...depending upon whether n is real ....

H. E. SALZER

Convair Astronautics
San Diego, California

DIETRICH SUSCHOWK, "Explicit formulae for 25 of the associated Legendre functions of the second kind," *MTAC*, v. 13, 1959, p. 303–305.

On p. 305, the first recurrence relation should read:

$$Q_n^{k+2}(x) = (n-k)(n+k+1)Q_n^{k}(x) - 2(k+1)\frac{x}{\sqrt{x^2-1}}Q_n^{k+1}(x)$$

JOHN HERNDON

Stanford Research Institute
Menlo Park, California

The SIAM Review

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Mathematical Methods in the Social Sciences, 1959, Proceedings of the First Stanford Symposium ..... Benoit Mandelbrot

Proceedings of an International Symposium on the Theory of Switching, Parts I and II ..... Raymond E. Miller

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